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Holonomy for gerbes over orbifolds

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Abstract

In this paper we compute explicit formulas for the holonomy map for a gerbe with connection over an orbifold. We show that the holonomy descends to a transgression map in Deligne cohomology. We prove that this recovers both the inner local systems in Ruan's theory of twisted orbifold cohomology [1] and the local system of Freed–Hopkins–Teleman in their work in twisted *K*-theory [2]. In the case in which the orbifold is simply a manifold we recover previous results of Gawędzki [3] and Brylinski[4].

Keywords: Gerbe; Holonomy; String theory; Deligne cohomology; Orbifold; Étale groupoid; Twisted *K*-theory

1. Introduction

A gerbe $\mathcal{L}$ over a manifold M (or a scheme, if you prefer) has much in common with a complex line bundle L.

A complex line bundle L is classified up to isomorphism class by a cohomology class $c_1(L) \in H^2(M;\mathbb{Z})$, its Chern class. By using the exponential sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{C}} \stackrel{\exp}{\to} \underline{\mathbb{C}}^\times \to 1$$

we can immediately interpret the Chern class of L as an element $[g]$ of the cohomology group $H^1(M;\underline{\mathbb{C}}^\times)$. In fact a Čech cocycle for this class is given by the gluing maps $g_{ij} : U_{ij} = U_i \cap U_j \to \mathbb{C}^\times$ of the line bundle for a Leray atlas $(U_i)_{i\in I}$ of the manifold M, namely one in which all open sets and their finite intersections are empty or contractible.

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Moreover, if we put a connection ∇ on the line bundle given locally by 1-form $A_i \in \Omega^1(U_i) \otimes \mathbb{C}$ then the curvature $F(L, \nabla) = \mathrm{d}A \in \Omega^2(M) \otimes \mathbb{C}$ satisfies the Bianchi identity

$$\mathrm{d}F = 0$$

and therefore it defines a cohomology class $[F] \in H^2(M; \mathbb{C})$. Weil [5] showed that $\left[\frac{F}{2\pi i}\right]$ is the image of $-c_1(L)$ under the map $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{C})$.

In any case $c_1(L)$ completely determines the isomorphism class $[L]$ of L—we say that $H^2(M; \mathbb{Z})$ is isomorphic to the group of isomorphism classes of line bundles over M. Later in this paper we will define a cohomology group $\mathbb{H}^2(M; \mathbb{Z}(2)^\infty_D)$ due to Deligne and Brylinski (and also Cheeger-Simons) that has the following properties:

- There is a surjective homomorphism $\mathbb{H}^2(M; \mathbb{Z}(2)^\infty_D) \to H^2(M; \mathbb{Z})$
- $\mathbb{H}^2(M; \mathbb{Z}(2)^\infty_D)$ classifies isomorphism classes $[L, \nabla]$ of line bundles with connection and the map above is realized by $[L, \nabla] \mapsto [L]$

Let us denote by $\mathcal{L}M$ the space of smooth maps from the circle S^1 to M (with no base point condition—$\mathcal{L}M$ is known as the *free loop space* of M). There is a tautological map

$$S^1 \times \mathcal{L}M \longrightarrow M$$

called the *evaluation map* sending $(z, \gamma) \mapsto \gamma(z)$. We can use this map together with the Künneth theorem and the fact that $H^1(S^1; \mathbb{Z}) = \mathbb{Z}$ to get

$$H^2(M; \mathbb{Z}) \to H^2(S^1 \times \mathcal{L}M; \mathbb{Z}) \cong H^2(\mathcal{L}M; \mathbb{Z}) \oplus (H^1(\mathcal{L}M; \mathbb{Z}) \otimes H^1(S^1; \mathbb{Z}))$$

$$\xrightarrow{\cong} H^2(\mathcal{L}M; \mathbb{Z}) \oplus H^1(\mathcal{L}M; \mathbb{Z}) \to H^1(\mathcal{L}M; \mathbb{Z}) \cong H^0(\mathcal{L}M; \underline{\mathbb{C}}^\times)$$

(where the next to last map is projection into the second component, and the last is induced by the exponential sequence). We call the resulting map $H^2(M; \mathbb{Z}) \to H^1(\mathcal{L}M; \mathbb{Z})$ the *transgression map.*

The previous discussion can be refined to get a map

$$\mathbb{H}^2(M; \mathbb{Z}(2)^\infty_D) \to H^0(\mathcal{L}M; \underline{\mathbb{C}}^\times)$$

that has a classical interpretation in terms of the *holonomy* of (L, ∇) along a closed path $\gamma \in \mathcal{L}M$. To wit, a connection ∇ on L produces a *parallel transport*, that is a linear map $P_{(L,\nabla)}(\gamma)$ for every path $\gamma : [a, b] \to M$ of the form

$$P_{(L,\nabla)}(\gamma) : L_{\gamma(a)} \to L_{\gamma(b)}$$

from the initial fiber to the final fiber.

Let us define the category $\mathcal{S}^0(M)$ that we call the *0-th Segal category of M*. Its objects are the points of M and its morphisms are paths $\gamma : [0, 2\pi] \to M$. We think of γ as an arrow from $\gamma(0)$ to $\gamma(2\pi)$. Then given a line bundle with connection (L, ∇) the parallel transport gives us a functor

$$P_{(L,\nabla)} : \mathcal{S}^0(M) \longrightarrow \textsf{Vector Spaces}$$

that assigns to the object $x \in M$ the one-dimensional vector space L_x and to the arrow γ the linear map $P_{(L,\nabla)}(\gamma)$.

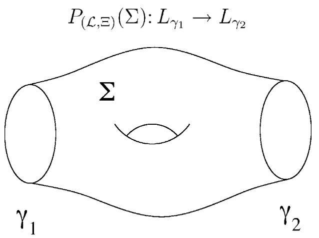
In particular, should γ be a closed path $\gamma : S^1 \to M$ then the linear isomorphism $P_{(L,\nabla)}(\gamma)$ can be canonically identified with an element of $\mathbb{C}^\times$, producing then a map

$$\mathcal{L}M \longrightarrow \mathbb{C}^\times$$

and hence and element in $H^0(\mathcal{L}M; \underline{\mathbb{C}}^\times)$.

Let us consider now a gerbe $\mathcal{L}$ over M. We will define gerbes later in the paper, but for now we list some of their properties.

- The group of isomorphism classes of gerbes $\mathsf{Gb}(M)$ on M is isomorphic to $H^3(M;\mathbb{Z})$.
- The isomorphism $\mathsf{Gb}(M) \to H^3(M;\mathbb{Z})$ is realized by the *Dixmier–Douady* characteristic class $dd(\mathcal{L}) \in H^3(M;\mathbb{Z})$ of the gerbe $\mathcal{L}$.
- We can place connections Ξ (also known as connective structures) on gerbes.
- The curvature of a connection over a gerbe $\mathcal{L}$ on M is a closed 3-form $G \in \Omega^3(M) \otimes \mathbb{C}$.
- The de Rham cohomology class $\left[\frac{G}{2\pi i}\right] \in H^3(M;\mathbb{R})$ is the real image of $dd(\mathcal{L})$.
- The group of isomorphism classes of gerbes with connections over M is isomorphic to a *Deligne cohomology* group $\mathbb{H}^3(M;\mathbb{Z}(3)^\infty_D)$.
- The holonomy of a gerbe $\mathcal{L}$ with connection Ξ is a complex line bundle L with connection ∇ on the free loop space $\mathcal{L}M$.
- The holonomy $(\mathcal{L}, \Xi) \mapsto (L, \nabla)$ realizes a transgression map
$$\mathbb{H}^3(M;\mathbb{Z}(3)^\infty_D) \to \mathbb{H}^2(\mathcal{L}M;\mathbb{Z}(2)^\infty_D)$$
- A pair $(\mathcal{L}, \Xi)$ induces a parallel transport functor
$$P_{(\mathcal{L},\Xi)} : \mathcal{S}^1(M) \longrightarrow \textsf{Vector Spaces}$$
from the first Segal category of M whose objects are maps from compact closed one-dimensional oriented manifolds to M, and whose morphisms are maps from compact two-dimensional manifolds to M forming cobordisms between two objects [6–8]. For instance, in the picture below we have two maps $\gamma_i : S^1 \to M$ $(i = 1, 2)$ and a map $\Sigma : F \to M$ from a two-dimensional manifold F into M. Such a configuration would produce a linear isomorphism



Such a functor is closely related to a *String Connection* in the terminology of Segal [8], Stolz, and Teichner [9]. More specifically if $\gamma : S^1 \to M$ is an object of $\mathcal{S}^1(M)$, then we have $P_{(\mathcal{L},\Xi)}(\gamma) = L_\gamma$, where L is the line bundle over $\mathcal{L}M$ mentioned in the last paragraph.

The purpose of this paper is to generalize the previous picture to the case in which instead of considering a manifold M we consider a smooth *orbifold* or *Deligne–Mumford stack* $\mathcal{X}$. This new case involves many new features and links together several interesting structures that have appeared in geometry and topology recently. We will briefly describe now the contents of this paper.

In Section 2 we set our notations and terminology for the theory of groupoids. We will use groupoids as models for our orbifolds —they will be our basic tool. We recall that a groupoid G is a category in which all morphisms have inverses and by an orbifold groupoid we mean a proper, smooth, étale groupoid. In this paper whenever we write groupoid a *smooth, étale groupoid* is to be understood.

In Section 2.1 we deal with the issue of defining the “free loop space" of an orbifold. Since an orbifold is already more than a space, the answer is itself an infinite dimensional orbifold that we call *the loop orbifold*. Our model for the loop orbifold will be a groupoid—the *loop groupoid* LG. In particular when the orbifold happens to be a manifold then the loop orbifold is simply the free loop space of the manifold.

In Section 2.2 we explain sheaf cohomology theory for groupoids. Then in Section 2.3 we use this theory to define Deligne cohomology for groupoids and explain its relation to the theory of n-gerbes with connective structure.

In Section 3 we prove the following theorem.

Theorem 1.1. *There is a natural transgression map* (*holonomy*)

$$\tau_1 : \check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_{\mathsf{G}}) \to \check{C}^0(\mathsf{LG}, \mathbb{C}^\times_{\mathsf{LG}})$$

that associates to every line bundle with connection over G *its holonomy. Here* $\mathbb{C}^\times_{\mathsf{LG}}$ *is the sheaf of* $\mathbb{C}^\times$ *valued functions on the loop groupoid. This map descends to cohomology*

$$\mathbb{H}^1(\mathsf{G}; \mathbb{Z}(1)^\infty_D) \to H^0\left(\mathsf{LG}, \mathbb{C}^\times_{\mathsf{LG}}\right).$$

In Section 4 we go ahead and *define* the Deligne cohomology for the loop groupoid to then prove the following theorem.

Theorem 1.2. *There is a natural holonomy homomorphism*

$$\tau_2 : \check{C}^2(\mathsf{G}, \mathbb{C}^\times(3)_{\mathsf{G}}) \to \check{C}^1(\mathsf{LG}, \mathbb{C}^\times(2)_{\mathsf{LG}})$$

from the group of gerbes with connection over G *to the group of line bundles with connection over the loop groupoid. Moreover this holonomy map commutes with the coboundary operator and therefore induces a map in orbifold Deligne cohomology*

$$\mathbb{H}^2(\mathsf{G}; \mathbb{Z}(2)^\infty_D) \to \mathbb{H}^1(\mathsf{G}; \mathbb{Z}(1)^\infty_D).$$

We prove in fact a little bit more. For what we really construct is a functor

$$\mathsf{LG} \to \mathsf{Vector\,Spaces}$$

given by parallel transport along arrows of the loop groupoid for the gerbe connection. While in the case of a manifold M this is only the portion of the string connection that associates the vector space to the objects of $\mathcal{S}^1(M)$, in the case of an orbifold groupoid G we already have arrow assignments. Since $\mathsf{LG} \hookrightarrow \mathcal{S}^1(\mathsf{G})$ is an inclusion of categories, we think of the functor constructed here as a genus-zero-one-input-one-output-ghost-part of the string connection. Indeed, the arrows of LG can be thought of as infinitely thin cylinders in the orbifold. This is an interesting issue. In fact, in the case in which one works with a global quotient orbifold (namely an orbifold that can be written with only one orbifold chart in the form $\mathsf{G} = [M/G]$ where M is a smooth manifold and G a *finite* group) and simultaneously with a *global* gerbe (namely a gerbe over $[M/G]$ whose data is given by global forms) we have worked out the explicit formulas for the full string connection, and not only the ghost part. The interested reader can find this computation in [10]. The general construction of the full string connection for $\mathcal{S}^1(\mathsf{G})$ would take us too far a field, we will return to this issue in a future paper.

Then in Section 4.1 we show that when the orbifold in question is simply a manifold we recover the results of Gawędzki [3] and Brylinski [4].

In Section 4.2 we study the particular case in which the orbifold X is actually a global quotient $[M/G]$, and the gerbe in question is globally defined.

Then in Section 4.3 we compute explicitly the case when the gerbe in question comes from discrete torsion in a global quotient orbifold $[M/G]$.

In Section 5 we pursue the subject of localization. One of the main results of [11] is the following theorem.

Theorem 1.3. *The fixed suborbifold of* LG *under the natural* S^1*-action* (*rotating the loops*) *is*

$$\wedge\mathsf{G} = (\mathsf{LG})^{S^1}$$

Here $\wedge\mathsf{G}$ is the so-called *twisted sector orbifold* or *inertia groupoid* of G. In the case of a manifold $\wedge M = M$.

We prove then the following theorem for G an orbifold groupoid

Theorem 1.4. *The restriction of the holonomy of a gerbe with connection over* G (*that is a line bundle with connection over* LG) *is an inner local system on* $\wedge\mathsf{G}$.

Inner local systems were discovered by Ruan [1] for completely different reasons. As it happens this is too the local system used by Freed, Hopkins and Teleman [2] in their work on twisted K-theory whenever the action of the Lie group is almost free, namely that it has only finite stabilizers.

Finally in Section 6 we discuss how to generalize the previous theory to n-gerbes with connection. The corresponding holonomy formula send $(n+1)$-gerbes to n-gerbes.

What we prove in this paper is actually a bit stronger than the statements of the previous theorems. We give *explicit formulas* for the holonomy maps, and then show that it descends to Deligne cohomology. Our motivation to do this is that in physics all the objects we have discussed have interesting interpretations and explicit formulas are necessary for computations [12]. For example we have explained elsewhere that in orbifold string theory (and conformal field theory) both the B-field and discrete torsion can be suitably interpreted in terms of gerbes and Deligne cohomology over orbifolds [13]. For related statements and work in the physics literature we refer the reader to [14] and the references therein.

2. Deligne cohomology for groupoids

When we say a groupoid we mean a (small) category G so that the set of its objects G_0 and the set of its arrows G_1 are both manifolds, and every arrow has an inverse.

We will represent orbifolds by groupoids. It is useful to consider the following two examples as motivation.

Example 2.1. Let G be a finite group. Consider the orbifold $[M/G]$ obtained from a G-manifold M (we use the brackets to differentiate the orbifold $[M/G]$ from the quotient space – or coarse moduli of orbits – M/G). Then we will associate to it the groupoid X with morphisms $\mathsf{X}_1 = M \times G$ and objects $\mathsf{X}_0 = M$. The arrow (m, g) takes the object m to the object mg. We will often write the groupoid X by the symbol $M \times G \rightrightarrows M$.

Example 2.2. Consider a manifold M with an atlas $\mathcal{U} = (U_i)_i$. We will associate to $(M, \mathcal{U})$ the groupoid $\mathsf{M}_{\mathcal{U}}$ whose objects $\mathsf{M}_0 = \{(x, i) : x \in U_i\} = \coprod_i U_i$ and whose arrows $\mathsf{M}_1 = \{(x, i, j) : x \in U_{ij} = U_i \cap U_j\} = \coprod_{(i,j)} U_{ij}$. The arrow (x, i, j) takes the object (x, i) to the object (x, j).

In the case of a general orbifold there is a groupoid that represents it that is a sort of hybrid of the previous two examples.

The groupoids G we will be concerned with will be étale and smooth, this means that all the structure maps (m = composition, i = inverse, e = identity, s = source, t = target) are local

$$\mathsf{G}_1\,{}_{\mathsf{t}}\!\times_{\mathsf{s}}\mathsf{G}_1 \xrightarrow{\mathsf{m}} \mathsf{G}_1 \xrightarrow{\mathsf{i}} \mathsf{G}_1 \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathsf{G}_0 \xrightarrow{\mathsf{e}} \mathsf{G}_1$$

diffeomorphisms, and the sets $\mathsf{G}_i = \mathsf{G}_1\ \mathsf{t}{\times}\mathsf{s}\ \mathsf{G}_1\ \mathsf{t}{\times}\mathsf{s}\ \cdots\ \mathsf{t}{\times}\mathsf{s}\ \mathsf{G}_1$ (I-times) are all manifolds. When the anchor map $(\mathsf{s},\mathsf{t}) : \mathsf{G}_1 \to \mathsf{G}_0 \times \mathsf{G}_0$ is proper the groupoid will represent an orbifold groupoid.

Definition 2.3. By an orbifold groupoid we mean smooth, étale, proper groupoid.

In this paper we are only concerned with smooth étale groupoids. When we think of a groupoid, implicitly what we are considering, is the Morita equivalence class where the groupoid belongs (see [15] for the definition of Morita equivalence). But in order to make calculations explicit, or to use Čech cohomology, we will make use of a special representative of the Morita class. We require this groupoid to be built out of a disjoint union of contractible sets as follows.

Definition 2.4. A groupoid G is called *Leray* if $\mathsf{G}_i = \mathsf{G}_1\ \mathsf{t}{\times}\mathsf{s}\ \mathsf{G}_1\ \mathsf{t}{\times}\mathsf{s}\ \cdots\ \mathsf{t}{\times}\mathsf{s}\ \mathsf{G}_1$ (i-times) is diffeomorphic to a disjoint union of contractible open sets for all $i \in \mathbb{N}$.

The existence of such Leray groupoid representative for every orbifold is proven by Moerdijk and Pronk [16, Cor. 1.2.5].

Here we are concerned with the geometry of the groupoid and we will give very explicit geometric descriptions of the objects in study. The algebraic topology of groupoids has been studied by several authors [17–19,15,16] and having both the geometric and the topological approaches is very useful.

Here we should introduce another very important structure associated to a groupoid called the inertia groupoid.

Definition 2.5. The inertia groupoid $\wedge\mathsf{G}$ is defined by:

- Objects $(\wedge\mathsf{G})_0$: Elements $v \in \mathsf{G}_1$ such that $\mathsf{s}(v) = \mathsf{t}(v)$.
- Morphisms $(\wedge\mathsf{G})_1$: For $v, w \in (\wedge\mathsf{G})_0$ an arrow $v \to^{\alpha} w$ is an element $\alpha \in \mathsf{G}_1$ such that $v \cdot \alpha = \alpha \cdot w$

$$\begin{tikzcd} \circ \arrow[loop left] \arrow[r, bend left, "\alpha"] & \circ \arrow[l, bend left, "\alpha^{-1}"] \arrow[loop right, "w^{-1}"] \end{tikzcd}$$

It is known that the inertia groupoid in the case of an orbifold matches with what is commonly known in the literature by twisted sectors (see [1,11]), thus this is a natural way to define them. And we will see in the next section that the inertia groupoid arises naturally as the constant loops of the loop groupoid.

2.1. Loop groupoid

In the last years it has become increasingly evident that the free loop space of a manifold $\mathcal{L}M = \mathrm{Map}(S^1, M)$ is a very important concept, providing a sort of natural thickening of M, it is at first not clear how one must define the free loop space of an orbifold. It is apparent that simply considering free loops in the quotient space $\mathsf{G}/{\sim}$ forgets all the orbifold structure. In fact the correct notion that we call the *loop orbifold* or *loop groupoid* is itself an orbifold, albeit an infinite dimensional one.

We have defined in [11] the loop groupoid to be a category whose objects are $\mathrm{Hom}(S^1, \mathsf{G})$ in the category of groupoids (its objects were known to Mrčun [20] and to Bridson-Haefliger [21]).

Example 2.6. Consider again the orbifold $X = [M/G]$ represented by the groupoid $M \times G \rightrightarrows M$ that we denote by X, as at the beginning of the last section. Consider the groupoid LX whose objects are all pairs (ϕ, g), $\phi : [0, 1] \to M$, $g \in G$ where we have $\phi(0)g = \phi(1)$. Let G act on the paths in the natural way, i.e. $\{\phi \cdot k\}(t) = \phi(t)k$ with $\{\phi \cdot k\}(0)k^{-1}gk = \{\phi \cdot k\}(1)$. We declare the arrows of LX to be the triples (ϕ, g, k) so that $\phi : [0, 1] \to M$, $\phi(0)g = \phi(1)$ and $k \in G$. The arrow $\Lambda = (\phi, g, k)$ in LX sends the path (ϕ, g) to $(\phi \cdot k, k^{-1}gk)$. We call LX *the loop groupoid*.

To do this in full generality we must face the following difficulty. Suppose that we first assign a groupoid to S^1 and consider the groupoid morphisms to G. They will certainly be a portion of the desired loop groupoid, but unfortunately we may be missing elements on it that will only become apparent by choosing a finer groupoid representation of the circle. Therefore we need to consider all the Morita equivalent groupoids representing the circle (this amounts to take finer and finer covers of the circle). This is explained in detail in [11]. The following formalism (that is unfortunately a bit technical) solves this difficulty.

For a finite set $\{q_1, \dots, q_n, q_0\} \subset (0, 1]$ with $q_1 < \cdots < q_n < q_0$ and $\epsilon > 0$ sufficiently small we associate a unique cover of the circle given by the sets $V_i^0 := (q_i - \epsilon, q_{i+1} + \epsilon)$ and the exponential map $e^{2\pi i t}$. This cover induces an *admissible cover* W on the real numbers $\mathbb{R}$ that consist of the sets $V_i^k := (q_i + k - \epsilon, q_{i+1} + k + \epsilon)$ and $V_0^k := (q_0 + k - 1 - \epsilon, q_1 + k + \epsilon)$ for $k \in \mathbb{Z}$ and $1 \leq i \leq n$. We call $\mathbb{R}^W$ the groupoid associated to this cover, i.e.

$$\mathbb{R}_1^W := \bigsqcup_{i,j,k,l} V_{i,j}^{k,l}, \qquad \mathbb{R}_0^W := \bigsqcup_{i,k} V_i^k$$

where $V_{i,j}^{k,l} := V_i^k \cap V_j^l$; and the ϵ is small enough so that all the double intersections are of the form $(q_i + k - \epsilon, q_i + k + \epsilon)$ or empty. We can now define the natural action of $\mathbb{Z}$ in $\mathbb{R}_1^W$

$$\mathbb{R}_1^W \times \mathbb{Z} \to \mathbb{R}_1^W, \qquad ((x, V_{i,j}^{k,l}), m) \to (x + m, V_{i,j}^{k+m,l+m})$$

with $x \in V_{i,j}^{k,l}$ and $x + m \in V_{i,j}^{k+m,l+m}$.

Definition 2.7. Let S_W^1 be the groupoid

$$\begin{array}{c} \mathbb{R}_1^W \times \mathbb{Z} \\ \downdownarrows \\ \mathbb{R}_0^W \end{array}$$

with maps

$$\mathsf{s}((x, V_{i,j}^{k,l}), m) = (x, V_i^k), \qquad \mathsf{t}((x, V_{i,j}^{k,l}), m) = (x + m, V_j^{l+m}),$$

$$\mathsf{e}(x, V_i^k) = ((x, V_{i,i}^{k,k}), 0), \qquad \mathsf{i}((x, V_{i,j}^{k,l}), m) = ((x + m, V_{j,i}^{l+m,k+m}), -m)$$

$$\mathsf{m}[((x, V_{i,j}^{k,l}, m), ((x + m, V_{i,j}^{k+m,l+m}), n)] = ((x, V_{i,j}^{k,l}), m + n).$$

The groupoid S^1_W is Morita equivalent to the unit groupoid $S^1 \rightrightarrows S^1$ (all arrows are identities). If W' is a refinement of W, then there is a unique Morita morphism $\rho_W^{W'} : \mathsf{S}^1_{W'} \to \mathsf{S}^1_W$.

Definition 2.8. For G a topological groupoid and an open cover W of the circle, the loop groupoid $\mathsf{LG}(W)$ associated to G and the open cover W will be defined by the following data:

- $\mathsf{LG}(W)_0$ the objects: morphisms of groupoids $\mathsf{S}^1_W \to \mathsf{G}$
- $\mathsf{LG}(W)_1$ the morphisms: for two elements in $\mathsf{LG}(W)_0$, say $\Psi, \Phi : \mathsf{S}^1_W \to \mathsf{G}$, a morphism (arrow) from Ψ to Φ is a map $\Lambda : \mathbb{R}_1^W \times \mathbb{Z} \to \mathsf{G}_1$ that makes the following diagram commute and such that for $r \in \mathbb{R}_1^W \times \mathbb{Z}$

$$\begin{tikzcd} \mathbb{R}_1^W \times \mathbb{Z} \arrow[r, "\Lambda"] \arrow[d, "\mathsf{s}\times\mathsf{t}"'] & \mathsf{G}_1 \arrow[d, "\mathsf{s}\times\mathsf{t}"] \\ \mathbb{R}_0^W \times \mathbb{R}_0^W \arrow[r, "(\Psi_0,\Phi_0)"'] & \mathsf{G}_0 \times \mathsf{G}_0 \end{tikzcd}$$

$$\Lambda(r) = \Psi_1(r) \cdot \Lambda(\mathsf{et}(r)) = \Lambda(\mathsf{es}(r)) \cdot \Phi_1(r). \tag{1}$$

The composition of morphisms is defined pointwise, in other words, for Λ and Ω with we set

$$\begin{tikzcd} \Psi \arrow[r, bend left, "\Lambda"] & \Phi \arrow[r, bend left, "\Omega"] & \Gamma \end{tikzcd}$$

$$\Omega \circ \Lambda(\mathsf{es}(r)) := \Lambda(\mathsf{es}(r)) \cdot \Omega(\mathsf{es}(r))$$

and

$$\Omega \circ \Lambda(r) := \Omega \circ \Lambda(\mathsf{es}(r)) \cdot \Gamma(r) = \Psi(r) \cdot \Omega \circ \Lambda(\mathsf{et}(r))$$

The previous properties imply that an arrow Λ determines its source Ψ and its target Φ. We consider $\mathsf{LG}(W)_1$ as a subspace of the space of smooth maps $\mathrm{Map}(\mathbb{R}_1^W, \mathsf{G}_1) \times \mathrm{Map}(\mathbb{R}_0^W, \mathsf{G}_1)$; in this way $\mathsf{LG}(W)_1$ and $\mathsf{LG}(W)_0$ inherit the compact-open topology, making the groupoid $\mathsf{LG}(W)$ into a topological one. For two admissible covers W_1, W_2 associated to $\{q_1, \dots, q_n, q_0\}$ and $\{q'_1, \dots, q'_{n'}, q'_0\}$ respectively, there is always a common refinement. We could take W associated to $\{q_1, \dots, q_n, q_0\} \cup \{q'_1, \dots, q'_{n'}, q'_0\}$ and the natural morphisms $\rho_{W_i}^W : \mathsf{S}^1_W \to \mathsf{S}^1_{W_i}$. These induce natural monomorphisms of topological groupoids $\mathsf{LG}(W_i) \hookrightarrow \mathsf{LG}(W)$. We want that two objects $\Psi_i : \mathsf{S}^1_{W_i} \to \mathsf{G}_1$ $(i = 1, 2)$ to be equivalent if the following square is commutative so we define

$$\begin{tikzcd} \mathsf{S}^1_W \arrow[r, "\rho^W_{W_1}"] \arrow[d, "\rho^W_{W_2}"'] & \mathsf{S}^1_{W_1} \arrow[d, "\Psi_1"] \\ \mathsf{S}^1_{W_2} \arrow[r, "\Psi_2"'] & \mathsf{G}. \end{tikzcd}$$

Definition 2.9. The loop groupoid LG of the groupoid G is defined as the monotone union (colimit) of the groupoids $\mathsf{LG}(W)$ where W runs over the set $\mathcal{W}$ of admissible covers

$$\mathsf{LG} := \varinjlim_{W \in \mathcal{W}} \mathsf{LG}(W).$$

In this way the loop groupoid is naturally endowed with a topology, becoming a topological groupoid. Now we list some facts about the loop groupoid that can be found in [11].

- For G étale and proper, then LG is also étale and has finite isotropy.
- A morphism of groupoids $\mathsf{F} \to \mathsf{G}$ induces naturally another one at the level of loops $\mathsf{LF} \to \mathsf{LG}$.
- If the morphism $\mathsf{F} \to \mathsf{G}$ is Morita, then $\mathsf{LF} \to \mathsf{LG}$ is also Morita.
- The loop groupoid LG can be endowed with an action of $\mathbb{R}$ in a natural way, i.e. shifting the morphisms by $t \in \mathbb{R}$. The fixed point set groupoid $\mathsf{LG}^{\mathbb{R}}$ under this action is Morita equivalent to the inertia groupoid $\wedge\mathsf{G}$ (Definition 2.5).

2.1.1. *The tangent loop groupoid*

The loop groupoid is endowed with a natural tangent groupoid in the same way the groupoid G is endowed with its tangent groupoid $T\mathsf{G}$ defined as $T\mathsf{G}_1 \rightrightarrows T\mathsf{G}_0$ with the induced structure maps (clearly $T\mathsf{G}$ is also smooth and étale).

Definition 2.10. For Ψ an object of $\mathsf{LG}(W)$, the tangent space $T_\Psi\mathsf{LG}(W)$ will consist of all morphisms $\xi : \mathsf{S}^1_W \to T\mathsf{G}$ such that $p \circ \xi = \Psi$ where $p : T\mathsf{G} \to \mathsf{G}$ is the natural projection morphism; these will be the objects of $T\mathsf{LG}(W)$. The morphisms of $T\mathsf{LG}(W)$ are the natural ones, namely for an arrow $\Lambda : \Psi \to \Phi$ with $\xi \in T_\Psi\mathsf{LG}(W)$ and $\zeta \in T_\Phi\mathsf{LG}(W)$, a tangent morphism in $T_\Lambda\mathsf{LG}(W)$ between ξ and ζ is a map $\nu : \mathbb{R}^1_W \times \mathbb{Z} \to T\mathsf{G}_1$ that makes the following diagram commute

$$\begin{tikzcd}
\mathbb{R}_1^W \times \mathbb{Z} \arrow[r, "\nu"'] \arrow[rr, bend left, "\Lambda"] \arrow[d, "\mathsf{s}\times\mathsf{t}"'] & T\mathsf{G}_1 \arrow[r, "p"'] \arrow[d, "\mathsf{s}\times\mathsf{t}"] & \mathsf{G}_1 \arrow[d, "\mathsf{s}\times\mathsf{t}"] \\
\mathbb{R}_0^W \times \mathbb{R}_0^W \arrow[r, "(\xi,\zeta)"] \arrow[rr, bend right, "(\Psi_0,\Phi_0)"'] & T\mathsf{G}_0 \times T\mathsf{G}_0 \arrow[r, "(p,p)"] & \mathsf{G}_0 \times \mathsf{G}_0
\end{tikzcd}$$

Taking the inverse limit over the admissible covers W of $T\mathsf{LG}(W)$ we obtain $T\mathsf{LG}$.

2.2. *Sheaves and cohomology*

All the properties of sheaves and cohomologies of topological spaces can be extended for the case of smooth étale groupoids. This is done in [22,17]. Let us briefly summarize the theory. A G-sheaf $\mathcal{F}$ is a sheaf over G_0, namely a topological space with a projection $p : \mathcal{F} \to \mathsf{G}_0$ which is a local homeomorphism on which G_1 acts continuously. This means that for $a \in \mathcal{F}_x = p^{-1}(x)$ and $g \in \mathsf{G}_1$ with $\mathsf{s}(g) = x$, there is an element ag in $\mathcal{F}_{t(g)}$ depending continuously on g and a. The action is a map $\mathcal{F}_p\times_s\mathsf{G}_1 \to \mathcal{F}$. For $\mathcal{F}$ a G-sheaf, a section $\sigma : \mathsf{G}_0 \to \mathcal{F}$ is called invariant if $\sigma(x)g = \sigma(y)$ for any arrow $x \to^g y$. $\Gamma_{\text{inv}}(\mathsf{G}, \mathcal{F})$ is the set of invariant sections and it will be an abelian group if $\mathcal{F}$ is an abelian sheaf. For an abelian G sheaf $\mathcal{F}$, the *cohomology* groups $H^n(\mathsf{G}, \mathcal{F})$ are defined as the cohomology groups of the complex:

$$\Gamma_{\text{inv}}(\mathsf{G}, \mathcal{I}^0) \to \Gamma_{\text{inv}}(\mathsf{G}, \mathcal{I}^1) \to \cdots$$

where $\mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ is a resolution of $\mathcal{F}$ by injective G-sheaves. When the abelian sheaf $\mathcal{F}$ is locally constant (for example $\mathcal{F} = \mathbb{Z}$) it is a result of Moerdijk [23] that

$$H^*(\mathsf{G}, \mathcal{F}) \cong H^*(B\mathsf{G}, \mathcal{F})$$

where the left hand side is sheaf cohomology and the right hand side is simplicial cohomology. There is a *basic spectral sequence* associated to this cohomology. Pulling back $\mathcal{F}$ along

$$\epsilon_n : \mathsf{G}_n \to \mathsf{G}_0 \tag{2}$$

$$\epsilon_n(g_1, \ldots, g_n) = \mathsf{t}(g_n)$$

it induces a sheaf $\epsilon_n^* \mathcal{F}$ on G_n (where the G action on G_n is the natural one, i.e. $(g_1, \ldots, g_n)h = (g_1, \ldots, g_n h)$; G_n becomes in this way a G-sheaf) such that for fixed q the groups $H^q(\mathsf{G}_p, \epsilon_p^* \mathcal{F})$ form a cosimplicial abelian group, inducing a spectral sequence:

$$H^p H^q(\mathsf{G}_\bullet, \mathcal{F}) \Rightarrow H^{p+q}(\mathsf{G}, \mathcal{F})$$

So if $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots$ is a resolution of G-sheaves with the property that $\epsilon_p^* \mathcal{F}^q$ is an acyclic sheaf on G_p, then $H^*(\mathsf{G}, \mathcal{F})$ can be computed from the double complex $\Gamma(\mathsf{G}_p, \epsilon_p^* \mathcal{F}^q)$. We conclude by introducing the algebraic gadget that will allow us to define Deligne cohomology. Let $\mathcal{F}^\bullet$ be a cochain complex of abelian sheaves, then the *hypercohomology groups* $\mathbb{H}^n(\mathsf{G}, \mathcal{F})$ are defined as the cohomology groups of the double complex $\Gamma_{\mathrm{inv}}(\mathsf{G}, \mathcal{I}^\bullet)$ where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism into a cochain complex of injectives.

2.3. Deligne cohomology

In what follows we will define the smooth Deligne cohomology of a smooth étale groupoid; we will extend the results of Brylinski[4] to groupoids and will follow very closely the description given in there. We will assume that G is Leray (Definition 2.4) and that the set of objects G_0 is of bounded cohomological dimension. Deligne cohomology is related to the De Rham cohomology. We will consider the De Rham complex of sheaves and we will truncate it at level p; what interests us is the degree p hypercohomology classes of this complex. To be more specific, let $\mathbb{Z}(p) := (2\pi\sqrt{-1})^p \cdot \mathbb{Z}$ be the cyclic subgroup of $\mathbb{C}$ and $\mathcal{A}^p_{\mathsf{G},\mathbb{C}}$ the G-sheaf of complex-valued differential p-forms; as G is a smooth étale groupoid the maps s and t are local diffeomorphisms, then the action of G into the sheaf over G_0 of complex-valued differential p-forms is the natural one given by the pull back of the corresponding diffeomorphism. Let $\mathbb{Z}(p)_{\mathsf{G}}$ be the constant $\mathbb{Z}(p)$-valued G-sheaf, and $i : \mathbb{Z}(p)_{\mathsf{G}} \to \mathcal{A}^0_{\mathsf{G},\mathbb{C}}$ the inclusion of constant into smooth functions.

Definition 2.11. Let G be a smooth étale groupoid. The **smooth Deligne complex** $\mathbb{Z}(p)^\infty_D$ is the complex of G-sheaves:

$$\mathbb{Z}(p)_{\mathsf{G}} \xrightarrow{i} \mathcal{A}^0_{\mathsf{G},\mathbb{C}} \xrightarrow{d} \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}$$

The hypercohomology groups $\mathbb{H}^q(\mathsf{G}, \mathbb{Z}(p)^\infty_D)$ are called the **smooth Deligne cohomology** of G.

In order to make the explanations clearer, where are going to work with a quasi-isomorphic complex of sheaves to the Deligne one, which is a bit simpler.

Definition 2.12. Let $\mathbb{C}^\times(p)_{\mathsf{G}}$ be the following complex of sheaves:

$$\mathbb{C}^\times_{\mathsf{G}} \stackrel{d\log}{\rightarrow} \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \stackrel{d}{\rightarrow} \cdots \stackrel{d}{\rightarrow} \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}$$

There is a quasi-isomorphism between the complexes $(2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{Z}(p)^\infty_D$ and $\mathbb{C}^\times(p)_{\mathsf{G}}[-1]$ (this fact is explained in Brylinski [4] p. 216)

$$\begin{tikzcd}
(2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{Z}(p)_{\mathsf{G}} \arrow[r] & \mathbb{C}_{\mathsf{G}} \arrow[r, "d"] \arrow[d, "\exp"'] & \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}} \arrow[d] \\
 & \mathbb{C}^\times_{\mathsf{G}} \arrow[r, "d\log"] & \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}
\end{tikzcd}$$

hence there is an isomorphism of hypercohomologies:

$$\mathbb{H}^{q-1}(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}) \cong (2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{H}^q(\mathsf{G}, \mathbb{Z}(p)^\infty_D) \tag{3}$$

Now let G be a Leray groupoid. We are going to define the Čech double complex associated to the G-sheaf complex $\mathbb{C}^\times(p)_{\mathsf{G}}$. Consider the space

$$C^{k,l} = \check{C}(\mathsf{G}_k, \mathcal{A}^l_{\mathsf{G},\mathbb{C}}) := \Gamma(\mathsf{G}_k, \epsilon_k^* \mathcal{A}^l_{\mathsf{G},\mathbb{C}})$$

of global sections of the sheaf $\epsilon_k^*\mathcal{A}^l_{\mathsf{G},\mathbb{C}}$ over G_k as in (2). The vertical differential $C^{k,l} \to C^{k,l+1}$ is given by the maps of the complex $\mathbb{C}^\times(p)_{\mathsf{G}}$ and the horizontal differential $C^{k,l} \to C^{k+1,l}$ is obtained by $\delta = \sum (-1)^i \delta_i$ where for $\sigma \in \Gamma(\mathsf{G}_k, \epsilon_k^*\mathcal{A}^l_{\mathsf{G},\mathbb{C}})$

$$(\delta_i\sigma)(g_1, \ldots, g_{k+1}) = \begin{cases} \sigma(g_1, \ldots, g_k) \cdot g_{k+1} & \text{for } i = k+1 \\ \sigma(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) & \text{for } 0 < i < k+1 \\ \sigma(g_2, \ldots, g_{k+1}) & \text{for } i = 0 \end{cases}$$

Definition 2.13. For G a smooth étale Leray groupoid, let us denote by $\check{C}^*(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}})$ the total complex

$$\check{C}^0(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}) \stackrel{\delta-d}{\rightarrow} \check{C}^1(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}) \stackrel{\delta+d}{\rightarrow} \check{C}^2(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}) \stackrel{\delta-d}{\rightarrow} \cdots$$

induced by the double complex with $(\delta + (-1)^i d)$ as coboundary operator. The Čech hypercoho-

$$\begin{tikzcd}
\vdots & \vdots & \vdots & & \vdots \\
\check{C}(\mathsf{G}_2, \mathbb{C}^\times_{\mathsf{G}}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_2, \mathcal{A}^1_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_2, \mathcal{A}^2_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \check{C}(\mathsf{G}_2, \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}) \arrow[u, "\delta"] \\
\check{C}(\mathsf{G}_1, \mathbb{C}^\times_{\mathsf{G}}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_1, \mathcal{A}^1_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_1, \mathcal{A}^2_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \check{C}(\mathsf{G}_1, \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}) \arrow[u, "\delta"] \\
\check{C}(\mathsf{G}_0, \mathbb{C}^\times_{\mathsf{G}}) \arrow[r, "d\log"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_0, \mathcal{A}^1_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \check{C}(\mathsf{G}_0, \mathcal{A}^2_{\mathsf{G},\mathbb{C}}) \arrow[r, "d"] \arrow[u, "\delta"] & \cdots \arrow[r, "d"] & \check{C}(\mathsf{G}_0, \mathcal{A}^{p-1}_{\mathsf{G},\mathbb{C}}) \arrow[u, "\delta"]
\end{tikzcd}$$

mology of the complex of sheaves $\mathbb{C}^\times(p)_{\mathsf{G}}$ is defined as the cohomology of the Čech complex $\check{C}(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}})$:

$$\check{H}^*(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}) := H^*\check{C}(\mathsf{G}, \mathbb{C}^\times(p)_{\mathsf{G}}).$$

As the G_i's are diffeomorphic to a disjoint union of contractible sets – Leray – then the previous cohomology actually matches the hypercohomology of the complex $\mathbb{C}^\times(p)_\mathsf{G}$, so we get

Lemma 2.14. *The cohomology of the Čech complex $\check{C}^*(\mathsf{G}, \mathbb{C}^\times(p)_\mathsf{G})$ is isomorphic to the hypercohomology of $\mathbb{C}^\times(p)_\mathsf{G}$*

$$\check{H}^*(\mathsf{G}, \mathbb{C}^\times(p)\mathsf{G}) \stackrel{\cong}{\to} \mathbb{H}^*(\mathsf{G}, \mathbb{C}^\times(p)_\mathsf{G}).$$

The Deligne cohomology groups classify the isomorphism classes of what is known as n-gerbes with connective structure.

Definition 2.15. An n-gerbe with connective structure over G is an $(n+1)$-cocycle of $\check{C}^{n+1}(\mathsf{G}, \mathbb{C}^\times(n+2)_\mathsf{G})$. Their isomorphism classes are classified by $\mathbb{H}^{n+1}(\mathsf{G}, \mathbb{C}^\times(n+2)_\mathsf{G}) = \mathbb{H}^{n+2}(\mathsf{G}, \mathbb{Z}(n+2)_D^\infty)$.

The following fact is more or less obvious, and relates this definition with the one given by the authors in [18]:

Proposition 2.16. *To have a* 1*-gerbe over* G *is the same thing as to have a line bundle* $\mathcal{L}$ *over* G_1 *together with maps* θ, h *satisfying the following properties*:

- $\mathsf{i}^*\mathcal{L} \cong^\theta \mathcal{L}^{-1}$
- $\pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L} \otimes \mathsf{m}^*\mathsf{i}^*\mathcal{L} \cong^h 1$
- $h : \mathsf{G}_1 \, {}_\mathsf{t}\!\times_\mathsf{s} \mathsf{G}_1 \to \mathbb{C}^\times$ is a 2-cocycle.

When the groupoid G is Leray, then the line bundle $\mathcal{L}$ is trivial and all the information is encoded in the 2-cocycle h. In this case a gerbe with connection will consist also of a 1-form $A \in \Omega^1(\mathsf{G}_1)$, a 2-form $B \in \Omega^2(\mathsf{G}_0)$ and a 3-form $K \in \Omega^3(\mathsf{G}_0)$ satisfying:

- $K = dB$
- $\mathsf{t}^*B - \mathsf{s}^*B = dA$ and
- $\pi_1^*A + \pi_2^*A - \mathsf{m}^*A = -\sqrt{-1}h^{-1}dh$

As we will see via the holonomy map:

- A 0-gerbe with connective structure induces a line bundle with connection over the groupoid G and a global 2-form on $\mathsf{G}/\sim$
- A 1-gerbe with connective structure induces what is known in the literature as a *gerbe with connection* over G and a global 3-form on $\mathsf{G}/\sim$.

Before finishing this section let us point out that the group $\mathbb{H}^{n-1}(\mathsf{G}, \mathbb{C}^\times(n)_\mathsf{G})$ is the only one that encodes really new information as the following proposition clarifies.

Proposition 2.17.

$$\mathbb{H}^p(\mathsf{G}, \mathbb{Z}(n)_D^\infty) \cong \mathbb{H}^{p-1}(\mathsf{G}, \mathbb{C}^\times(n)_\mathsf{G}) = \begin{cases} H^{p-1}(\mathsf{G}, \underline{\mathbb{C}}^\times) = H^p(\mathsf{G}, \mathbb{Z}) & \text{for } p > n \\ H^{p-1}(\mathsf{G}, \mathbb{C}^\times) & \text{for } p < n \end{cases}$$

where $\underline{\mathbb{C}}^\times$ *stands for the sheaf of* $\mathbb{C}^\times$ *valued functions.*

Proof. Let us have a look at the double complex of Definition 2.13. When $p > n$ the p-cocycles are over the diagonal, so the information of all the columns of the double complex besides the first is irrelevant. This is because the sheaves $\mathcal{A}^i_{\mathsf{G},\mathbb{C}}$ are acyclic. Now, when $p < n$ and $(h, \omega_1, \dots, \omega_{p-1})$ is a $(p-1)$-cocycle, by a successive application of Poincaré lemma it is possible to find an element $(f, \theta_1, \dots, \theta_{p-2})$ in $\check{C}^{p-2}(\mathsf{G}, \mathbb{C}^\times(n)_\mathsf{G})$ such that

$$(h, \omega_1, \dots, \omega_{p-1}) - (\delta + (-1)^{p-2} d)(f, \theta_1, \dots, \theta_{p-2}) = (h - \delta f, 0, \dots, 0)$$

with $d(h - \delta f) = 0$, a locally constant $\mathbb{C}^\times$ function. This implies the second isomorphism. □

After this brief summary of definitions we are ready to define the holonomy map for **smooth étale groupoids**.

3. Holonomy

In the same way that a line bundle with connection over a manifold M induces a $\mathbb{C}^\times$ valued function on the free loop space of M, given by the holonomy around a loop, we can define its analogous to smooth étale groupoids. Let us recall that the groupoid in mind is Leray, so we can make use of the Čech description of the hypercohomology.

Let W be an admissible cover of the circle associated to the set $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ with $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ as in Section 2.1.

Theorem 3.1. *There is a natural transgression map* (*holonomy*)

$$\tau_1 : \check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_\mathsf{G}) \to \check{C}^0(\mathsf{LG}(W), \mathbb{C}^\times_{\mathsf{LG}(W)})$$

that sends cocycles to cocycles and that descends to cohomology

$$\mathbb{H}^1\left(\mathsf{G}, \mathbb{C}^\times_\mathsf{G} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{G},\mathbb{C}}\right) \to H^0\left(\mathsf{LG}(W), \mathbb{C}^\times_{\mathsf{LG}(W)}\right).$$

Proof. First we will set up the notation. The pair (h, A) will be an element in $\check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_\mathsf{G})$ with $h : \mathsf{G}_1 \to \mathbb{C}^\times$ and $A \in \Gamma(\mathsf{G}_0, \mathcal{A}^1_{\mathsf{G},\mathbb{C}})$. The object $\psi : \mathsf{S}^1_W \to \mathsf{G}$ of the loop groupoid $\mathsf{LG}(W)$ will consist of maps $\psi_i : I_i = [\alpha_{i-1}, \alpha_i] \to \mathsf{G}_0$ and arrows $\psi : \{\alpha_1, \dots, \alpha_n\} \to \mathsf{G}_1$ such that

$$\mathsf{s}(\psi(\alpha_i)) = \psi_i(\alpha_i) \quad \text{and} \quad \mathsf{t}(\psi(\alpha_i)) = \psi_{i+1}(\alpha_i)$$

and when $i = n \mathsf{t}\psi(\alpha_n) = \psi_1(\alpha_0)$.

So $\tau_1(h, A)$ defines a function $H : \mathsf{LG}(W)_0 \to \mathbb{C}^\times$ as follows:

$$H(\psi) := \exp\left(\sum_{i=1}^{n} \int_{I_i} \psi_i^* A\right) \prod_{i=1}^{n} h(\psi(\alpha_i))^{-1}. \tag{4}$$

It is clearly an homomorphism. We show now that H descends to cohomology. Suppose that (h, A) is a 1-cocycle, its coboundary $(d + \delta)(h, A)$ is zero, i.e.

$$\mathsf{t}^* A - \mathsf{s}^* A = -d \log h, \quad \text{in } \mathsf{G}_1 \tag{5}$$

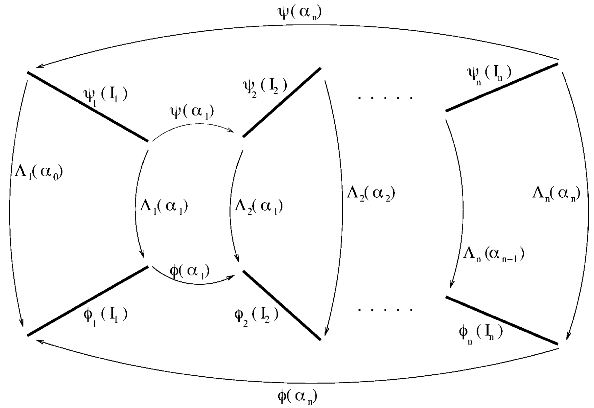
$$h(g_2) h(g_1 g_2)^{-1} h(g_1) = 1 \quad \text{for } (g_1, g_2) \in \mathsf{G}_2, \tag{6}$$

We want to see that the coboundary δ of $\tau_1(h, A) = H$ is also zero. The cycle δH is a function $\mathsf{LG}(W)_1 \to \mathbb{C}^\times$ that for the arrow Λ between ψ and ϕ takes the value $\delta H(\Lambda) = H(\phi) H(\psi)^{-1}$.

The arrow Λ will consist of maps $\Lambda_i : I_i \to \mathsf{G}_1$ such that

$$\Lambda_i(\alpha_i) \cdot \phi(\alpha_i) = \psi(\alpha_i) \cdot \Lambda_{i+1}(\alpha_i)$$

where $\Lambda_{n+1}(\alpha_n) := \Lambda_0(\alpha_0)$. In the following diagram the dark lines are the images of the intervals in G_0 and the arrows are elements in G_1.



(7)

Making use of the property (5) of the 1-cocycle (h, A) we get the following set of equalities:

$$\frac{\exp\left(\int_{I_i} \phi_i^* A\right)}{\exp\left(\int_{I_i} \psi_i^* A\right)} = \exp\left(\int_{I_i} \Lambda_i^*(\mathsf{t}^* A - \mathsf{s}^* A)\right) = \exp\left(\int_{I_i} \Lambda_i^*(-d\log h)\right) = \frac{h(\Lambda_i(\alpha_{i-1}))}{h(\Lambda_i(\alpha_i))},$$

and using property (6) we have

$$\delta H(\Lambda) = \frac{H(\phi)}{H(\psi)} = \prod_{i=1}^{n} \frac{h(\Lambda_i(\alpha_{i-1}))}{h(\Lambda_i(\alpha_i))} \frac{h(\psi(\alpha_i))}{h(\phi(\alpha_i))} = \prod_{i=1}^{n} \frac{h(\psi(\alpha_i))h(\Lambda_{i+1}(\alpha_i))}{h(\Lambda_i(\alpha_i))h(\phi(\alpha_i))}$$

$$= \prod_{i=1}^{n} \frac{h(\Lambda_i(\alpha_i) \cdot \phi(\alpha_i))}{h(\psi(\alpha_i) \cdot \Lambda_{i+1}(\alpha_i))} = 1$$

This means that H is invariant under the action of $\mathsf{LG}(W)_1$ therefore it defines a map

$$H : \mathsf{LG}(W)/\sim \to \mathbb{C}^\times$$

Now if (h, A) is a coboundary, i.e. $(h, A) = (\delta f, -d\log f)$ for some $f : \mathsf{G}_0 \to \mathbb{C}^\times$ then $H = \tau_1(\delta f, -d\log f)$ will become

$$H(\psi) = \exp\left(\sum_{i=1}^{n} \int_{I_i} \psi_i^*(-d\log f)\right) \prod_{i=1}^{n} \delta f(\psi(\alpha_i))^{-1}$$

$$= \prod_{i=1}^{n} \frac{f(\psi_i(\alpha_{i-1}))}{f(\psi_i(\alpha_i))} \prod_{n=1}^{n} \frac{f(\mathsf{s}(\psi(\alpha_i)))}{f(\mathsf{t}(\psi(\alpha_i)))} = 1.$$

Hence the map τ_1 descends to cohomology. □

Another way to understand the previous result is the following. The pair (h, A) represents a complex line bundle with connection (L, Δ) over the groupoid G. The function $\tau_1(h, A)$ assigns a complex number to every element ψ in the loop groupoid. This number represents an endomorphism of the fiber L_ψ obtained through the parallel transport given by the connection Δ.

If we now take a refinement W' of the cover W associated to the set $\{\alpha_0, \ldots, \alpha_{i-1}, \beta, \alpha_i, \ldots, \alpha_n\}$, $\rho_W^{W'} : \mathsf{S}^1_{W'} \to \mathsf{S}^1_W$ the natural morphism and $\psi : \mathsf{S}^1_W \to \mathsf{G}$ a loop, we can see that for $\psi' := \psi \circ \rho_W^{W'}$ the equality $H(\psi) = H(\psi')$ holds. This because the morphism $\psi(\beta)$ is equal to $\mathsf{e}\psi_i(\beta)$ and by property 6 the function h restricted to $\mathsf{e}(\mathsf{G}_0)$ is constant and equal to 1. Then the function H is stable under cover refinement, and therefore we can take the inverse limits on the admissible covers and to obtain the holonomy morphism over the loop groupoid LG:

Proposition 3.2. *There is a natural transgression map* (*holonomy*)

$$\tau_1 : \check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_{\mathsf{G}}) \to \check{C}^0(\mathsf{LG}, \mathbb{C}^\times_{\mathsf{LG}})$$

that sends cocycles to cocycles and that descends to cohomology

$$\mathbb{H}^1\left(\mathsf{G}, \mathbb{C}^\times_{\mathsf{G}} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{G},\mathbb{C}}\right) \to H^0(\mathsf{LG}, \mathbb{C}^\times_{\mathsf{LG}}).$$

Remark 3.3. Using the definition of the holonomy given by Brylinski [4] in Lemma 6.1.2 and taking G to be a Leray groupoid naturally associated to a manifold M as in Example 2.2 (i.e. G is built out of a open contractible cover of M) we see that the previous map matches the holonomy of a connection in a line bundle around a loop in M.

4. The Line bundle over the loop groupoid

From a gerbe with connection over the groupoid G we are going to construct a line bundle over the loop groupoid, in a way that is compatible with the transgression map on a manifold. The main result of this section is:

Theorem 4.1. *There is a natural homomorphism*

$$\tau_2 : \check{C}^2(\mathsf{G}, \mathbb{C}^\times(3)_{\mathsf{G}}) \to \check{C}^1(\mathsf{LG}(W), \mathbb{C}^\times(2)_{\mathsf{LG}(W)})$$

that sends 2-cocycles to 1-cocycles (*i.e. gerbes with connection over* G *to line bundles with connection over the loop groupoid* LG), *commutes with the coboundary operator* (*i.e.* $\tau_2 \circ (\delta + d) = (\delta - d) \circ \tau_1$) *and therefore induces a map in cohomology*

$$\mathbb{H}^2\left(\mathsf{G}, \mathbb{C}^\times_{\mathsf{G}} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \stackrel{d}{\to} \mathcal{A}^2_{\mathsf{G},\mathbb{C}}\right) \to \mathbb{H}^1\left(\mathsf{LG}(W), \mathbb{C}^\times_{\mathsf{LG}(W)} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{LG}(W),\mathbb{C}}\right).$$

Proof. Let us first fix the notation. The triple (h, A, B) will be an element of $\check{C}^2(\mathsf{G}, \mathbb{C}^\times(3)_{\mathsf{G}})$ with $h : \mathsf{G}_2 \to \mathbb{C}^\times$, $A \in \Gamma(\mathsf{G}_1, \mathcal{A}^1_{\mathsf{G},\mathbb{C}})$ and $B \in \Gamma(\mathsf{G}_0, \mathcal{A}^2_{\mathsf{G},\mathbb{C}})$. The arrow Λ of the loop groupoid $\mathsf{LG}(W)$ between the objects ψ and ϕ will be defined as in Theorem 3.1. The arrow ν of the tangent loop groupoid $T_\Lambda\mathsf{LG}(W)$ between the objects $\xi \in T_\psi\mathsf{LG}(W)$ and $\zeta \in T_\phi\mathsf{LG}(W)$ (as in definition 2.10) will consist of maps $\xi_i, \zeta_i : I_i \to T\mathsf{G}_0$, $\nu_i : I_i \to T\mathsf{G}_1$ and $\xi, \zeta : \{\alpha_1, \ldots, \alpha_n\} \to T\mathsf{G}_1$ such that:

$$\mathsf{s}(\xi_i(\alpha_i)) = \xi_i(\alpha_i), \qquad \mathsf{t}(\xi(\alpha_i)) = \xi_{i+1}(\alpha_i), \qquad \mathsf{s}(\zeta(\alpha_i)) = \zeta_i(\alpha_i),$$
$$\mathsf{t}(\zeta(\alpha_i)) = \zeta_{i+1}(\alpha_i), \qquad \nu_i(\alpha_i) \cdot \zeta(\alpha_i) = \xi(\alpha_i) \cdot \nu_{i+1}(\alpha_i)$$

and when $i = n$ $\mathsf{t}\xi(\alpha_n) = \xi_1(\alpha_0)$, $\mathsf{t}\zeta(\alpha_n) = \zeta_1(\alpha_0)$ and $\nu_{n+1}(\alpha_n) := \nu_0(\alpha_0)$. □

Now we are ready to define $\tau_2(h, A, B)$. It will consist of the pair $(F, \Delta) \in \check{C}^1(\mathsf{LG}(W), \mathbb{C}^\times(1)_{\mathsf{LG}(W)})$ with $F : \mathsf{LG}(W) \to \mathbb{C}^\times$ a map, and $\Delta : T\mathsf{LG}(W)_0 \to \mathbb{C}$ a linear functional on the tangent loop space.

Definition 4.2.

$$F(\Lambda) := \exp\left(\sum_{i=1}^{n} \int_{I_i} \Lambda_i^* A\right) \prod_{i=1}^{n} \frac{h(\psi(\alpha_i), \Lambda_{i+1}(\alpha_i))}{h(\Lambda_i(\alpha_i), \phi(\alpha_i))},$$

$$\langle \Delta_\psi, \xi\rangle := \sum_{i=1}^{n} \int_{I_i} B\left(\frac{\mathrm{d}\psi_i}{\mathrm{d}t}, \xi_i(t)\right) \mathrm{d}t + \sum_{i=1}^{n} \langle A_{\psi(\alpha_i)}, \xi(\alpha_i)\rangle$$

In this way the map τ_2 is clearly an homomorphism.

The other two statements of the theorem will be proven separately.

Proposition 4.3. *τ_2 sends cocycles to cocycles.*

Proof. In other words we need to prove that $(\delta - d)(h, A, B) = 0$ implies $(\delta + d)(F, \Delta) = 0$, i.e. a gerbe with connection over the groupoid induces a line bundle with connection over the loop groupoid.

The cocycle condition $(\delta h, \delta A - d\log h, \delta B - \mathrm{d}A) = 0$ implies:

$$h(a, b)h(a, bc)^{-1}h(ab, c)h(b, c)^{-1} = 1 \quad \text{for}\,(a, b, c) \in \mathsf{G}_3 \tag{8}$$

$$\pi_2^* A + \pi_1^* A - \mathsf{m}^* A = d\log h \,\text{in}\, \mathsf{G}_2 \tag{9}$$

$$\mathsf{t}^* B - \mathsf{s}^* B = dA \,\text{in}\, \mathsf{G}_1. \tag{10}$$

Let us prove first that $\delta F = 1$. This in particular implies that the map $F : \mathsf{LG}(W) \to \mathbb{C}^\times$ is a morphism of groupoids.

Let Λ and Ω be two arrows in the loop groupoid with $\psi \to^{\Lambda} \phi \to^{\Omega} \gamma$, we need to calculate $\delta F(\Lambda, \Omega) = F(\Lambda)F(\Omega)F(\Lambda \cdot \Omega)^{-1}$.

Using the property (9) we have that

$$\exp\left(\int_{I_i} \Lambda_i^* A + \Omega_i^* A - (\Lambda \cdot \Omega)_i^* A\right) = \frac{h(\Lambda_i(\alpha_i), \Omega_i(\alpha_i))}{h(\Lambda_i(\alpha_{i-1}), \Omega_i(\alpha_{i-1}))}$$

and by applying property (8) to the triples

$$(\psi(\alpha_i), \Lambda_{i+1}(\alpha_i), \Omega_{i+1}(\alpha_i)) \quad \text{and} \quad (\Lambda_i(\alpha_i), \Omega_i(\alpha_i), \gamma(\alpha_i))$$

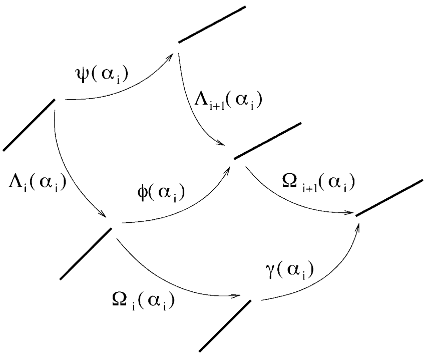
we get

$$\frac{h(\psi(\alpha_i), \Lambda_{i+1}(\alpha_i))}{h(\Lambda_i(\alpha_i), \phi(\alpha_i))} \frac{h(\phi(\alpha_i), \Omega_{i+1}(\alpha_i))}{h(\Omega_i(\alpha_i), \gamma(\alpha_i))} \left(\frac{h(\psi(\alpha_i), (\Lambda \cdot \Omega)_{i+1}(\alpha_i))}{h((\Lambda \cdot \Omega)_i(\alpha_i), \gamma(\alpha_i))}\right)^{-1}$$
$$= \frac{h(\Lambda_{i+1}(\alpha_i), \Omega_{i+1}(\alpha_i))}{h(\Lambda_i(\alpha_i), \Omega_i(\alpha_i))}$$

Multiplying the last two equations and making the product over the i's it follows that $\delta F(\Lambda, \Omega) = 1$.

Now we will prove that $(\delta\Delta + d\log F) = 0$. Let $\nu \in T_\Lambda \mathsf{LG}(W)$, we want to check

$$\langle -(d\log F)_\Lambda, \nu\rangle = \langle (\delta\Delta)_\Lambda, \nu\rangle$$



and we will do so by integrating over a 1-parameter thickening of the path Λ in he direction of ν. Via the tubular neighborhood diffeomorphism and the fact that ν and Λ are determined by a finite number of maps over compact sets, we can find a one-parameter family $\Lambda^s \in \mathsf{LG}(W)$, with $s \in [-\epsilon, \epsilon]$ for ϵ sufficiently small, such that $\Lambda = \Lambda^0$ and $\frac{\mathrm{d}\Lambda^s}{\mathrm{d}s} = \nu$. We claim that

$$\int_{-\epsilon}^{\epsilon} \left\langle -(d\log F)_{\Lambda^s}, \frac{\mathrm{d}\Lambda^s}{\mathrm{d}s} \right\rangle \mathrm{d}s = \int_{-\epsilon}^{\epsilon} \left\langle (\delta\Delta)_{\Lambda^s}, \frac{\mathrm{d}\Lambda^s}{\mathrm{d}s} \right\rangle \mathrm{d}s$$

Let us first elaborate on the left hand side (LHS). The steps will be outlined after the set of equalities.

$$\mathrm{LHS} = -\sum_{i=1}^{n} \left(\int_{I_i} (\Lambda_i^{\epsilon})^* A - (\Lambda_i^{-\epsilon})^* A \right) \tag{11}$$

$$- \int_{-\epsilon}^{\epsilon} \left\langle \mathrm{d}\log \prod_{i=1}^{n} \frac{h(\psi^s(\alpha_i), \Lambda_{i+1}^s(\alpha_i))}{h(\Lambda_i^s(\alpha_i), \phi^s(\alpha_i))}, \frac{\mathrm{d}\Lambda^s}{\mathrm{d}s} \right\rangle \mathrm{d}s \tag{12}$$

$$= -\sum_{i=1}^{n} \left(\int_{I_i} \left\langle A_{\Lambda_i^{\epsilon}(t)}, \frac{\mathrm{d}\Lambda_i^{\epsilon}}{\mathrm{d}t} \right\rangle - \left\langle A_{\Lambda_i^{-\epsilon}(t)}, \frac{\mathrm{d}\Lambda_i^{-\epsilon}}{\mathrm{d}t} \right\rangle \mathrm{d}t \right) \tag{13}$$

$$-\sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \left(\left\langle A_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\phi^s(\alpha_i)}, \frac{\mathrm{d}\phi^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right. \tag{14}$$

$$\left. + \left\langle A_{\Lambda_{i+1}^s(\alpha_i)}, \frac{\mathrm{d}\Lambda_{i+1}^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\Lambda_i^s(\alpha_i)}, \frac{\mathrm{d}\Lambda_i^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right) \mathrm{d}s \tag{15}$$

Line (11) is obtained after evaluating the integral at the end points $-\epsilon$ and ϵ. Line (13) is the same as line (11) but written in a different way, and lines (14) and (15) are obtained from line (12) after using property (9) and the fact that

$$\psi^s(\alpha_i) \cdot \Lambda_{i+1}^s(\alpha_i) = \Lambda_i^s(\alpha_i) \cdot \phi^s(\alpha_i).$$

For the right hand side we need to make use of Stoke's theorem.

$$\mathrm{RHS} = \int_{-\epsilon}^{\epsilon} \left\langle \Delta_{\phi_i^s}, \frac{d\phi_i^s}{ds} \right\rangle - \left\langle \Delta_{\psi_i^s}, \frac{\mathrm{d}\psi_i^s}{\mathrm{d}s} \right\rangle \mathrm{d}s \tag{16}$$

$$= \int_{-\epsilon}^{\epsilon} \left(\sum_{i=1}^{n} \int_{I_i} B\left(\frac{\mathrm{d}\phi_i^s}{\mathrm{d}t}, \frac{\mathrm{d}\phi_i^s}{\mathrm{d}s}\right) - B\left(\frac{\mathrm{d}\psi_i^s}{\mathrm{d}t}, \frac{\mathrm{d}\psi_i^s}{\mathrm{d}s}\right) \mathrm{d}t \right) \mathrm{d}s \tag{17}$$

$$+ \sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \left(\left\langle A_{\phi^s(\alpha_i)}, \frac{\mathrm{d}\phi^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right) \mathrm{d}s \tag{18}$$

$$= \sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \int_{I_i} \mathrm{d}A \left(\frac{\mathrm{d}\Lambda_i^s}{\mathrm{d}t}, \frac{d\Lambda_i^s}{\mathrm{d}s} \right) \mathrm{d}t\, \mathrm{d}s \tag{19}$$

$$+ \sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \left(\left\langle A_{\phi^s(\alpha_i)}, \frac{\mathrm{d}\phi^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right) \mathrm{d}s \tag{20}$$

$$= \sum_{i=1}^{n} \left(\int_{I_i} \left\langle A_{\Lambda_i^{-\epsilon}(t)}, \frac{\mathrm{d}\Lambda_i^{-\epsilon}}{\mathrm{d}t} \right\rangle - \left\langle A_{\Lambda_i^{\epsilon}(t)}, \frac{\mathrm{d}\Lambda_i^{\epsilon}}{\mathrm{d}t} \right\rangle \mathrm{d}t \right) \tag{21}$$

$$+ \sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \left(\left\langle A_{\Lambda_i^s(\alpha_i)}, \frac{\mathrm{d}\Lambda_i^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\Lambda_{i+1}^s(\alpha_i)}, \frac{\mathrm{d}\Lambda_{i+1}^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right) \mathrm{d}s \tag{22}$$

$$+ \sum_{i=1}^{n} \int_{-\epsilon}^{\epsilon} \left(\left\langle A_{\phi^s(\alpha_i)}, \frac{\mathrm{d}\phi^s}{\mathrm{d}s}(\alpha_i) \right\rangle - \left\langle A_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i) \right\rangle \right) \mathrm{d}s \tag{23}$$

Line (16) is obtained after applying the coboundary operator to Δ. Expanding via the definition of Δ we get lines (17) and (18). From property (10) we get line (19) from (17). And lines (20) and (21) come from line (19) and Stoke's theorem (evaluating A at the boundary).

Lines (13), (14) and (15) match lines (21), (23) and (22), respectively. Therefore LHS = RHS and the pair (F, Δ) is a 1-cocycle. □

The second part of the theorem states

Proposition 4.4. *The map τ_2 commutes with the coboundary operator* (i.e. $\tau_2 \circ (\delta + d) = (\delta - d) \circ \tau_1$)

Proof. Let the pair (f, G) be in $\check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_{\mathsf{G}})$, $H := \tau_1(f, G)$, $(\delta - d)H = (\delta H, -d \log H)$, $(\delta + d)(f, G) = (\delta f, \delta G + d \log f, dG)$ and $(F, \Delta) := \tau_2(\delta f, \delta G + d \log f, dG)$. We want to prove that

$$(\delta H, -d \log H) = (F, \Delta).$$

Replacing in $F(\Lambda)$ we get:

$$\begin{aligned} F(\Lambda) &= \exp\left(\sum_{i=1}^{n} \int_{I_i} \Lambda_i^*(\delta G + d \log f) \right) \prod_{i=1}^{n} \frac{\delta f(\psi(\alpha_i), \Lambda_{i+1}(\alpha_i))}{\delta f(\Lambda_i(\alpha_i), \phi(\alpha_i))} \\ &= \exp\left(\sum_{i=1}^{n} \int_{I_i} \phi^* G - \psi^* G \right) \prod_{i=1}^{n} \frac{f(\Lambda_i(\alpha_i))}{f(\Lambda_i(\alpha_{i-1}))} \prod_{i=1}^{n} \frac{f(\psi(\alpha_i)) f(\Lambda_{i+1}(\alpha_i))}{f(\Lambda_i(\alpha_i)) f(\phi(\alpha_i))} \\ &= \exp\left(\sum_{i=1}^{n} \int_{I_i} \phi^* G - \psi^* G \right) \frac{f(\psi(\alpha_i)))}{f(\phi(\alpha_i))} = H(\phi) H(\psi)^{-1} = \delta H(\Lambda) \end{aligned}$$

To prove $-d\log H = \Delta$ we proceed as in the previous proposition. We claim that

$$\int_{-\epsilon}^{\epsilon}\left\langle -(d\log H)_{\psi^s}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}\right\rangle \mathrm{d}s = \int_{-\epsilon}^{\epsilon}\left\langle (\Delta)_{\psi^s}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}\right\rangle \mathrm{d}s.$$

Elaborating on each side in the same way it was done for the previous proposition we have that

$$\mathrm{LHS} = -\sum_{i=1}^{n}\left(\int_{I_i}(\psi_i^{\epsilon})^*G - (\psi_i^{-\epsilon})^*G\right) \tag{24}$$

$$-\sum_{i=1}^{n}\int_{-\epsilon}^{\epsilon}\left\langle (d\log f^{-1})_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i)\right\rangle \mathrm{d}s \tag{25}$$

$$\mathrm{RHS} = \sum_{i=1}^{n}\int_{-\epsilon}^{\epsilon}\int_{I_i} dG\left(\frac{\mathrm{d}\psi^s}{\mathrm{d}t}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}\right)\mathrm{d}t\,\mathrm{d}s \tag{26}$$

$$\sum_{i=1}\int_{-\epsilon}^{\epsilon}\left\langle (\delta G + d\log f)_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i)\right\rangle \mathrm{d}s \tag{27}$$

$$= \sum_{i=1}^{n}\left(\int_{I_i}(\psi_i^{-\epsilon})^*G - (\psi_i^{\epsilon})^*G\right) \tag{28}$$

$$+\sum_{i=1}^{n}\int_{-\epsilon}^{\epsilon}\left\langle G_{\psi_i^s(\alpha_i)}, \frac{\mathrm{d}\psi_i^s}{\mathrm{d}s}(\alpha_i)\right\rangle - \left\langle G_{\psi_i^s(\alpha_{i-1})}, \frac{\mathrm{d}\psi_i^s}{\mathrm{d}s}(\alpha_{i-1})\right\rangle \mathrm{d}s \tag{29}$$

$$+\sum_{i=1}^{n}\int_{-\epsilon}^{\epsilon}\left\langle G_{\psi_{i+1}^s(\alpha_i)}, \frac{\mathrm{d}\psi_{i+1}^s}{\mathrm{d}s}(\alpha_i)\right\rangle - \left\langle G_{\psi_i^s(\alpha_i)}, \frac{\mathrm{d}\psi_i^s}{\mathrm{d}s}(\alpha_i)\right\rangle \mathrm{d}s \tag{30}$$

$$+\sum_{i=1}^{n}\int_{-\epsilon}^{\epsilon}\left\langle (d\log f)_{\psi^s(\alpha_i)}, \frac{\mathrm{d}\psi^s}{\mathrm{d}s}(\alpha_i)\right\rangle \mathrm{d}s \tag{31}$$

where the lines (24)–(27) are obtained after replacing the given information and lines (28) and (29) come from (26) and Stoke's theorem. Lines (29) and (30) are equal with opposite signs; therefore we have that LHS = RHS.

So we have that the following square is commutative:

$$\begin{tikzcd}
\check{C}^1(\mathsf{G}, \mathbb{C}^\times(2)_{\mathsf{G}}) \arrow[r, "\tau_1"] \arrow[d, "(\delta+d)"'] & \check{C}^0(\mathsf{LG}(W), \mathbb{C}^\times_{\mathsf{LG}(W)}) \arrow[d, "(\delta-d)"] \\
\check{C}^2(\mathsf{G}, \mathbb{C}^\times(3)_{\mathsf{G}}) \arrow[r, "\tau_2"'] & \check{C}^1(\mathsf{LG}(W), \mathbb{C}^\times(2)_{\mathsf{LG}(W)})
\end{tikzcd}$$

With the previous two propositions it is clear that τ_2 descends to a map in cohomology and Theorem 4.1 follows. □

Taking the limit over the admissible covers we obtain the following statement.

Corollary 4.5. *There is a natural homomorphism*

$$\tau_2 : \check{C}^2(\mathsf{G}, \mathbb{C}^\times(3)_{\mathsf{G}}) \to \check{C}^1(\mathsf{LG}, \mathbb{C}^\times(2)_{\mathsf{LG}})$$

that sends 2*-cocycles to 1-cocycles* (*i.e. gerbes with connection over* G *to line bundles with connection over the loop groupoid*), *commutes with the coboundary operator* (*i.e.* $\tau_2 \circ (\delta + d) = (\delta - d) \circ \tau_1$) *and therefore induces a map in cohomology*

$$\mathbb{H}^2\left(\mathsf{G}, \mathbb{C}^\times_{\mathsf{G}} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{G},\mathbb{C}} \stackrel{d}{\to} \mathcal{A}^2_{\mathsf{G},\mathbb{C}}\right) \to \mathbb{H}^1\left(\mathsf{LG}, \mathbb{C}^\times_{\mathsf{LG}} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathsf{LG},\mathbb{C}}\right).$$

4.1. Manifolds

In the case that the groupoid M represents a manifold M via a good open cover $\{U_\alpha\}_{\alpha\in I}$ (by good we mean that the finite intersections $U_{\alpha_1\ldots\alpha_n} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ are either empty or contractible—namely a Leray cover) with

$$\mathsf{M}_0 = \bigsqcup_{\alpha\in I} U_\alpha \quad \text{and} \quad \mathsf{M}_1 = \bigsqcup_{(\alpha,\beta)\in I^2} U_{\alpha\beta}$$

we obtain the construction introduced by Gawędzki of a line bundle with connective structure over $\mathcal{L}M$ via a gerbe with connection over M [3, p. 108–113](it can also be found in Brylinski's book [4, Proposition 6.5.1]). The information of the cocycle (h, A, B) whose cohomology class lies on the group $\mathbb{H}^2\left(\mathsf{M}, \mathbb{C}^\times_{\mathsf{M}} \to^{d\log} \mathcal{A}^1_{\mathsf{M},\mathbb{C}} \to^{d} \mathcal{A}^2_{\mathsf{M},\mathbb{C}}\right)$ is equivalent to the data

$$h_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to \mathbb{C}^\times, \qquad A_{\alpha\beta} \in \Omega^2(U_{\alpha\beta}) \otimes \mathbb{C}, \quad \text{and} \quad B_\alpha \in \Omega^1(U_\alpha) \otimes \mathbb{C}$$

so that

$$h_{\alpha\beta\gamma} h^{-1}_{\alpha\beta\delta} h_{\alpha\gamma\delta} h^{-1}_{\beta\gamma\delta} = 1 \quad \text{in } U_{\alpha\beta\gamma\delta}, \qquad A_{\alpha\beta} + A_{\beta\gamma} - A_{\alpha\gamma} = d\log h_{\alpha\beta\gamma} \quad \text{in } U_{\alpha\beta\gamma},$$
$$B_\beta - B_\alpha = dA_{\alpha\beta} \quad \text{in } U_{\alpha\beta}.$$

Let $\Lambda : \psi \to \phi$ be an arrow in the loop groupoid $\mathsf{LM}(W)$. Recall that as M is a manifold the category M does not have automorphisms besides the identity (i.e. there is only one arrow from a point to itself) . Define the indices $\kappa_i, \lambda_i \in I$ such that

$$\psi_i(I_i) \subset U_{\kappa_i} \qquad \phi_i(I_i) \subset U_{\lambda_i},$$

and therefore $\Lambda(I_i) \subset U_{\kappa_i\lambda_i}$. Hence, the formula (8) can be written as:

$$F(\Lambda) = \exp\left(\sum_{i=1}^n \int_{I_i} \Lambda_i^* A_{\kappa_i\lambda_i}\right) \prod_{i=1}^n \frac{h_{\kappa_i\kappa_{i+1}\lambda_{i+1}}(\Lambda(\alpha_i))}{h_{\kappa_i\lambda_i\lambda_{i+1}}(\Lambda(\alpha_i))}$$

with $\kappa_{n+1} = \kappa_1$ and $\lambda_{n+1} = \lambda_1$. If $\xi \in T_\psi\mathsf{LM}$ is a vector field over ψ (a tangent vector of the LM), then the formula (8) can be written as:

$$(\Delta, \xi)_\psi = \sum_{i=1}^n \int_{I_i} B_{\kappa_i}\left(\frac{\mathrm{d}\psi_i}{\mathrm{d}t}, \xi_i(t)\right) \mathrm{d}t + \sum_{i=1}^n \langle A_{\kappa_i\kappa_{i+1}}(\psi(\alpha_i)), \xi(\alpha_i)\rangle.$$

This assignment matches the ones given by Gawędzki [3, p. 111]and Brylinski [4, p. 250]. As $\mathcal{L}M$, the loop space of M, and the loop groupoid LM are Morita equivalent (see [11, Proposition 5.1.3]) we can deduce Brylinski's result:

Proposition 4.6 ([4, Proposition 6.5.1]). *The assignment $(h, A, B) \mapsto (F, \Delta)$ induces a group homomorphisms*

$$\mathbb{H}^2\left(M, \mathbb{C}^\times_M \stackrel{d\log}{\to} \mathcal{A}^1_{M,\mathbb{C}} \stackrel{d}{\to} \mathcal{A}^2_{M,\mathbb{C}}\right) \to \mathbb{H}^1\left(\mathcal{L}M, \mathbb{C}^\times_{\mathcal{L}M} \stackrel{d\log}{\to} \mathcal{A}^1_{\mathcal{L}M,\mathbb{C}}\right)$$

and is equal to the opposite of the transgression map from sheaves of groupoids with connective structure and curving over M to line bundles with connection over $\mathcal{L}M$.

4.2. Global quotients

For the purpose of illustration let us consider an orbifold of the form $[M/G]$ obtained from a manifold in which G acts as a finite subgroup of Diff(M). Moreover let us assume that the gerbe (h, A, B) that we consider can be represented on the groupoid X with morphisms $\mathsf{X}_1 = M \times G$ and objects $\mathsf{X}_0 = M$, where the arrow (m, g) takes the object m to the object mg. Namely:

- $B \in \Omega^2(M) \otimes \mathbb{C}$
- $A \in \Omega^1(M \times G) \otimes \mathbb{C}$
- $h : \mathsf{X}_2 = M \times G \times G \to \mathbb{C}^\times$,

and writing $A_g := A|_{M\times\{g\}}$ and $h_{g,k} := h|_{M\times\{g\}\times\{k\}}$ for $g, k \in G$,we have

$$g^*B - B = dA_g, \qquad A_g + A_k + A_{(gk)^{-1}} = d\log h_{g,h}.$$

Let $\Lambda = (\phi, g, k)$ be an arrow of the loop groupoid LX (as explained in Example 2.6) from (ϕ, g) to $(\phi \cdot k, k^{-1}gk)$ where $\phi : [0, 1] \to M$ and $\phi(0)g = \phi(1)$. Then the formula (8) can be written in this case as:

$$F(\Lambda) = \exp\left(\int_0^1 \phi^* A_k\right) \frac{h_{g,k}(\phi(0))}{h_{k,k^{-1}gk}(\phi(0))}.$$

And if ξ is a vector field along ϕ (i.e. $\xi \in \Gamma([0, 1], \phi^*TM)$) with $\xi(0)g = \xi(1)$ then the functional of equation (8) can be expressed as:

$$(\Delta, \xi)_\phi = \int_0^1 B\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}, \xi(t)\right) \mathrm{d}t + \langle A_g(\phi(0)), \xi(0)\rangle$$

4.3. Discrete torsion

When considering conformal field theories on an orbifold $[M/G]$ it is well-known that for any non-trivial cocycle $\varepsilon : G \times G \to \mathbb{C}^\times$ and $[\alpha] \in H^2(G, \mathbb{C}^\times)$, a new model can be defined by weighting the twisted sectors of the orbifold with a non-trivial phase, the so-called discrete torsion (see [24]). As argued in [13] this can be seen as a choice of flat B-field over the target stack $[M/G]$.

Let X be a groupoid associated to $[M/G]$, and as in the previous section take $\mathsf{X}_0 := M$ and $\mathsf{X}_1 := M \times G$ with the natural source and target maps. Let $\bar{G} := * \times G \rightrightarrows *$ be the natural groupoid representative of G. The morphism $\mathsf{X} \to \bar{G}$ induces a monomorphism $H^2(G, \mathbb{C}^\times) \to \mathbb{H}^2(\mathsf{X}, \mathbb{C}^\times(3)_\mathsf{X})$ that allows to define a flat gerbe as follows

$$h : \mathsf{X}_2 = M \times G \times G \to \mathbb{C}^\times, \qquad (x, g_1, g_2) \mapsto \varepsilon(g_1, g_2), \qquad A = B = 0.$$

As the flat gerbe only depends on the group G and not on the geometry of M, we need not to work with an open cover of the orbifold X. We know [11, Proposition 6.1.1]that the loop groupoid is Morita equivalent to the groupoid

$$\mathsf{LX} = \begin{array}{c} \left(\bigsqcup_g \mathcal{P}_g\right) \times G \\ \downdownarrows \\ \left(\bigsqcup_g \mathcal{P}_g\right) \end{array}$$

where $\mathcal{P}_g := \{\phi : [0,1] \to M | \phi(0)g = \phi(1)\}$ and G acts on the paths in the natural way, i.e. $\{\phi \cdot k\}(t) = \phi(t)k$ with $\{\phi \cdot k\}(0)k^{-1}gk = \{\phi \cdot k\}(1)$.

For an arrow $\Lambda = (\phi, g, k)$ in LX between the paths (ϕ, g) and $(\phi \cdot k, k^{-1}gk)$ with $x = \phi(0)$, the morphism of groupoids F becomes:

$$F : \mathsf{LX} \to \mathbb{C}^\times, \qquad (\phi, g, k) \mapsto \frac{h\left((x,g),(xg,k)\right)}{h\left((x,k)(xk,k^{-1}gk)\right)} = \frac{\varepsilon(g,k)}{\varepsilon(k,k^{-1}gk)}$$

and the connection Δ is equal to zero.

In this way we obtain a flat line bundle over the loop groupoid LX that once restricted to the inertia groupoid produces the discrete torsion. This localization procedure is explained in the next section.

5. Localization at the fixed points

In [11] we argued that the inertia groupoid can be understood as the fixed point set of the action of the real numbers over the loops (the action shifts the paths by a real number). This groupoid has for objects the constant paths i.e. maps $\psi : \mathbb{R} \to \mathsf{G}_0$ with $\psi(t) = x \in \mathsf{G}_0$ for all t, and for morphisms constant arrows $\Lambda : \mathbb{R} \to \mathsf{G}_1$. This description is equivalent to the one given in the Definition 2.5.

As we need to remember the source and the target of the morphisms, the elements of $(\wedge\mathsf{G})_1$ will be pairs $(v, \alpha) \in \mathsf{G}_2$ such that $v \in \wedge\mathsf{G}_0$, $s(v,\alpha) = v$ and $t(v,\alpha) = \alpha^{-1}v\alpha$. The structure maps of $\wedge\mathsf{G}$ will be written with the letters s, t, e, i, m to differentiate them from the ones of G.

Hence we have an inclusion of groupoids $j : \wedge\mathsf{G} \to \mathsf{LG}$ and so we can pull back the line bundle (F, Δ) previously described to obtain a line bundle with connection over the inertia groupoid.

Lemma 5.1. *The line bundle* $(f, \omega) := j^*(F, \Delta) = (F, \Delta)|_{\wedge\mathsf{G}}$ *over the inertia groupoid* $\wedge\mathsf{G}$ *is flat.*

Proof. As the paths representing $\wedge\mathsf{G}$ are constant, is easy to see that

$$f = F|_{\wedge\mathsf{G}} \quad \text{and} \quad \omega = \Delta|_{\wedge\mathsf{G}_0} = A|_{\wedge\mathsf{G}_0}.$$

From Eq. (10) we see that $\mathrm{d}d\omega = 0$ because the maps s and t in $\wedge\mathsf{G}_0$ are equal. Then the connection ω over $\wedge\mathsf{G}$ is flat. □

These line bundles are the representatives on what Ruan has coined "inner local systems" (see [1]) which he uses to twist the Chen-Ruan cohomology of orbifolds. In fact, all the constructions he has of "inner local systems" could be done using the procedure outlined in this paper. We believe the only relevant local systems are the ones obtained via transgression from a gerbe with connection.

Definition 5.2. In our terminology an "inner local system" is a flat line bundle $\mathcal{L}$ over the inertia groupoid $\wedge\mathsf{G}$ such that:

- $\mathcal{L}$ is trivial once restricted to $\mathsf{e}(\mathsf{G}_0) \subset \wedge\mathsf{G}_1$ (i.e. $\mathcal{L}|_{\mathsf{e}(\mathsf{G}_0)} = 1$) and
- $i^*\mathcal{L} = \mathcal{L}^{-1}$ where $i : \wedge\mathsf{G} \to \wedge\mathsf{G}$ is the inverse map (i.e. $(i(v, \alpha) = (\alpha^{-1} v\alpha, \alpha^{-1})))$.

There is an extra condition in Ruan's definition that is trivially fulfilled by $\mathcal{L}$. It just says that if $f : \wedge\mathsf{G} \to \mathbb{C}^\times$ is the map that contains the information on transition functions, then $f(\alpha_1)f(\alpha_2) = f(\alpha_1\alpha_2)$ for composable morphisms; this is true because f is a morphism of groupoids.

Proposition 5.3. *The line bundle* (f, ω) *over* $\wedge\mathsf{G}$ *is an inner local system for* G.

Proof. As the paths of $\wedge\mathsf{G}$ are constant, from the Eq. (8) we see that

$$f(v, \alpha) = \frac{h(v, \alpha)}{h(\alpha, \alpha^{-1} v\alpha)} \quad \text{for}\,(v, \alpha) \in \wedge\mathsf{G}_1.$$

If $v = \mathsf{e}(x)$ and α goes from x to y then $f(\mathsf{e}(x), \alpha) = f(\mathsf{e}(x), \mathsf{e}(x))$ and $f(\alpha, \mathsf{e}(y)) = f(\mathsf{e}(y), \mathsf{e}(y))$; this follows from the cocycle condition of f applied to the triples $(\mathsf{e}(x), \mathsf{e}(x), \alpha)$ and $(\alpha, \mathsf{e}(y), \mathsf{e}(y))$. Hence $f(v, \alpha) = \frac{h(\mathsf{e}(x), \mathsf{e}(x))}{h(\mathsf{e}(y), \mathsf{e}(y))}$, which means that the value of the gluing functions do not depend on the arrow but on its end points. This implies that the restriction of $\mathcal{L}$ to $\mathsf{e}(\mathsf{G}_0)$ is trivial.

Now as f is a morphism of groupoids, then $f(\Lambda)^{-1} = f(i\Lambda)$ and hence the second condition holds. □

5.1. Global quotients

Recall that for the orbifold $\mathsf{X} := [M/G]$ the inertia groupoid $\wedge\mathsf{X}$ is Morita equivalent to $\sqcup_{(g)}[M^g/C(g)]$ where M^g are the fixed point set of g, $C(g)$ is the centralizer of g in G and the disjoint union runs over (g) the conjugacy classes of elements in G.

If we forget the connective structure, the construction outlined in this paper assigns to every gerbe over X a line bundle over $\wedge\mathsf{X}$. Via this transgression we get $C(g)$ equivariant line bundles $\mathcal{L}_g$ over M^g.

$$H^3_G(M, \mathbb{Z}) \cong H^3(\mathsf{X}, \mathbb{Z}) \longrightarrow H^2(\wedge\mathsf{X}, \mathbb{Z}) \cong \bigoplus_{(g)} H^2_{C(g)}(M^g, \mathbb{Z})$$

These line bundles form an inner local system in the sense of Ruan, but also they are the coefficients Freed–Hopkins–Teleman [2] used to twist the cohomology of the twisted sectors in order to get a Chern character isomorphism with the twisted K-theory of the orbifold.

In the case of a gerbe coming from discrete torsion we obtain $\mathrm{U}(1)$ representations of the groups $C(g)$. These representations were used by Adem and Ruan [25] to twist the orbifold cohomology and in this way they obtained an isomorphism with the twisted orbifold K-theory [18].

6. Generalized holonomy

In this last section we want to emphasize that the holonomy map for gerbes over a groupoid can be generalized to n-gerbes.

Theorem 6.1. *There is a natural homomorphism*

$$\tau_n : \check{C}^n(\mathsf{G}, \mathbb{C}^\times(n+1)_{\mathsf{G}}) \to \check{C}^{n-1}(\mathsf{LG}, \mathbb{C}^\times(n)_{\mathsf{LG}})$$

that sends n-cocycles to $(n-1)$*-cocycles* (*i.e.* $(n-1)$ *gerbes with connection over* G *to* $(n-2)$ *gerbes with connection over the loop groupoid*), *commutes with the coboundary operator* (*i.e.* $\tau_n \circ (\delta + (-1)^n d) = (\delta + (-1)^{n-1} d) \circ \tau_{n-1}$) *and therefore induces a map in cohomology*

$$\mathbb{H}^n(\mathsf{G}, \mathbb{C}^\times(n+1)_{\mathsf{G}}) \to \mathbb{H}^{n-1}(\mathsf{LG}, \mathbb{C}^\times(n)_{\mathsf{LG}}).$$

Proof. Let us define first the map τ_n. Take $(\omega, \theta^1, \dots, \theta^n) \in \check{C}^n(\mathsf{G}, \mathbb{C}^\times(n+1)_{\mathsf{G}})$ with $\omega : \mathsf{G}_n \to \mathbb{C}^\times$ and $\theta^j \in \Gamma(\mathsf{G}_{n-j}, \mathcal{A}^j_{\mathsf{G},\mathbb{C}})$ and let

$$(F, \Delta^1, \dots, \Delta^{n-1}) := \tau_n(\omega, \theta^1, \dots, \theta^n)$$

with $F : \mathsf{LG}_{n-1} \to \mathbb{C}^\times$ and $\Delta^j \in \Gamma(\mathsf{LG}_{n-1-j}, \mathcal{A}^j_{\mathsf{LG},\mathbb{C}})$ defined in the following way.

For $\Lambda = (\Lambda^1, \dots, \Lambda^{n-1})$ a set of $n-1$ composable morphisms in LG_{n-1} joining the objects $\psi^0, \dots, \psi^{n-1}$ with $\Lambda_i : I_i = [\alpha_{i-1}, \alpha_i] \to \mathsf{G}_{n-1}$ for $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$, we define

$$F(\mathbf{\Lambda}) := \exp\left(\sum_{i=1}^{p} \int_{I_i} (\Lambda_i)^* \theta^1\right) \times \prod_{i=1}^{p} \prod_{j=0}^{n-1} \times (\omega(\Lambda_i^1(\alpha_i), \dots, \Lambda_i^j(\alpha_i), \psi^j(\alpha_i), \Lambda_{i+1}^{j+1}(\alpha_i), \dots, \Lambda_{i+1}^{n-1}(\alpha_i)))^{(-1)^{j+n}}$$

Now let $\mathbf{\Xi^a} = (\Xi^{a,1}, \dots, \Xi^{a,n-1-k})$, $\mathbf{a} \in \{\mathbf{1}, \dots, \mathbf{k}\}$ be vector fields over $\mathbf{\Lambda} = (\Lambda^1, \dots, \Lambda^{n-1-k}) \in \mathsf{LG}_{n-1-k}$ with $\mathbf{\Xi}^a{}_i : I_i \to (T\mathsf{G})_{n-1-k}$, joining the objects $\xi^{a,0}, \dots, \xi^{a,n-k-1}$ of the tangent loop groupoid; i.e. $\xi^{a,j}$ is a vector field over ψ^j and $\Xi^{a,j}$ is an arrow between $\xi^{a,j-1}$ and $\xi^{a,j}$ as well as a vector field over Λ^j.

For $m \in \{0, \dots, n-1-k\}$ we construct the following set of arrows in $(T\mathsf{G})_{n-k}$:

$$\vartheta_m \mathbf{\Xi}^a(\alpha_i) := \left(\Xi_i^{a,1}(\alpha_i), \dots, \Xi_i^{a,m}(\alpha_i), \xi^m(\alpha_i), \Xi_{i+1}^{a,m+1}(\alpha_i), \dots, \Xi_{i+1}^{a,n-1-k}(\alpha_i)\right).$$

Define,

$$\langle \Delta^k_{\mathbf{\Lambda}}, (\Xi^{\mathbf{1}}, \dots, \Xi^{\mathbf{k}}) \rangle := \sum_{i=1}^{p} \int_{I_i} \theta^{k+1} \left(\frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}t}, \mathbf{\Xi}_i^{\mathbf{1}}(t), \dots, \mathbf{\Xi}_i^{\mathbf{k}}(t)\right) \mathrm{d}t + \sum_{i=1}^{p} \sum_{m=0}^{n-1-k} (-1)^{m+n} \langle \theta^k, (\vartheta_m \mathbf{\Xi}^{\mathbf{1}}(\alpha_i), \dots, \vartheta_m \mathbf{\Xi}^{\mathbf{k}}(\alpha_i)) \rangle$$

In what follows we will only show that the map τ_n sends cocycles to cocycles. The other part of the proof can be done following the steps of Theorem 4.1. Let us suppose that $\delta\theta^{k+1} = (-1)^n \,\mathrm{d}\theta^k$ and we want to prove that $\delta\Delta^{k+1} = (-1)^{n-1} d\Delta^k$.

Both $\delta\Delta^{k+1}$ and $\mathrm{d}\Delta^k$ are in $\check{C}(\mathsf{LG}_{n-1-k}, \mathcal{A}^{k+1}_{\mathsf{LG},\mathbb{C}})$, so we need to take $\mathbf{\Xi^a}$ $\mathbf{a} \in \{\mathbf{1}, \dots \mathbf{k+1}\}$ vector fields over Λ.

The proof of

$$\left\langle \delta\Delta^{k+1}, \left(\mathbf{\Xi}^{\mathbf{1}}, \dots, \mathbf{\Xi}^{\mathbf{k+1}}\right)\right\rangle = (-1)^{n-1}\left\langle d\Delta^k, \left(\mathbf{\Xi}^{\mathbf{1}}, \dots, \mathbf{\Xi}^{\mathbf{k+1}}\right)\right\rangle$$

will be done by thickening Λ in the directions of the $\Xi^{\mathbf{a}}$ and then integrating over this tubular neighborhood.

Then let $\mathbf{\Lambda}(\vec{s}) \in \mathsf{LG}_{n-1-k}$ with $\vec{s} := (s_1, \ldots, s_{k+1})$ be such that $\mathbf{\Lambda}(\vec{0}) = \mathbf{\Lambda}$ and $\frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}|_{\vec{s}=0} = \Xi^{\mathbf{a}}$. We argue that

$$\int_{[-\epsilon,\epsilon]^{k+1}} \left\langle \delta\Delta^{k+1}, \left(\frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_1}, \ldots, \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_{k+1}}\right)\right\rangle \mathrm{d}\vec{s}$$
$$= (-1)^{n-1} \int_{[-\epsilon,\epsilon]^{k+1}} \left\langle \mathrm{d}\Delta^{k}, \left(\frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_1}, \ldots, \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_{k+1}}\right)\right\rangle \mathrm{d}\vec{s}. \tag{32}$$

We just need one last piece of information, the face maps associated to the coboundary operator δ. They are $\varrho_l : \mathsf{G}_{n-k} \to \mathsf{G}_{n-k-1}$ with $\varrho_l(g_1, \ldots, g_{n-k}) = (g_1, \ldots, g_l g_{l+1}, \ldots, g_{n-k})$, $\varrho_0(g_1, \ldots, g_{n-k}) = (g_2, \ldots, g_{n-k})$, $\varrho_{n-k}(g_1, \ldots, g_{n-k}) = (g_1, \ldots, g_{n-k-1})$ and $\rho_l : \mathsf{LG}_{n-k-1} \to \mathsf{LG}_{n-k-2}$ defined in the same way.

It is easy to see that

$$\varrho_l\left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right) = \vartheta_{m-1}\left(\rho_l \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}\right)(\alpha_i) \quad \text{for } l < m,$$
$$\varrho_{m-1}\left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right) = \varrho_{m-1}\left(\vartheta_{m-1} \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right),$$
$$\varrho_l\left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right) = \vartheta_{m}\left(\rho_{l-1} \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}\right)(\alpha_i) \quad \text{for } l > m+1$$

and note that the only elements not paired are

$$\varrho_0\left(\vartheta_0 \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right) \quad \text{and} \quad \varrho_{n-k}\left(\vartheta_{n-k-1} \frac{\mathrm{d}\mathbf{\Lambda}(\vec{s})}{\mathrm{d}s_a}(\alpha_i)\right),$$

these will play an important role in what follows.

Writing $\mathbf{\Lambda} := \mathbf{\Lambda}(\vec{s})$ we have that the left hand side of (32) becomes:

$$\begin{aligned}\mathrm{LHS}(32) = \int_{[-\epsilon,\epsilon]^{k+1}} &\left(\sum_{i=1}^{p} \int_{I_i} (-1)^n \,\mathrm{d}\theta^{k+1} \left(\frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_1}, \ldots, \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_{k+1}}\right) \mathrm{d}t\right.\\ &+ \sum_{i=1}^{p} \sum_{m=0}^{n-k-2} \sum_{l=0}^{n-k-1} (-1)^{m+l+n}\\ &\times \left.\left\langle \theta^{k+1}; \vartheta_m\left(\rho_l \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_1}\right)(\alpha_i), \ldots, \vartheta_m\left(\rho_l \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_{k+1}}\right)(\alpha_i)\right\rangle\right) \mathrm{d}\vec{s}\end{aligned}$$

after replacing $\delta\theta^{k+2}$ by $(-1)^n\,\mathrm{d}\theta^{k+1}$, and composing by the maps ρ_l from the definition of δ.

And the right hand side becomes:

$$\begin{aligned}\mathrm{RHS}(32) = &\sum_{j=1}^{k+1} (-1)^{j+n} \sum_{i=1}^{p} \int_{[-\epsilon,\epsilon]^{k}} \int_{I_i} \theta^{k+1} \left(\frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}t}, \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_1}, \ldots, \widehat{\frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_j}}, \ldots, \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_{k+1}}\right)\Bigg|_{s_j=-\epsilon}^{s_j=\epsilon}\\ &\times \mathrm{d}t\,\mathrm{d}\vec{s} + \int_{[-\epsilon,\epsilon]^{k+1}} \sum_{i=1}^{p} \sum_{m=0}^{n-1-k} \sum_{l=0}^{n-k} (-1)^{m+l+3n-1}\end{aligned}$$

$$\times \left\langle \theta^{k+1}; \varrho_l \left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}s_1}(\alpha_i) \right), \ldots, \varrho_l \left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}s_{k+1}}(\alpha_i) \right) \right\rangle \mathrm{d}\vec{s}$$

after evaluating in the boundary of $[-\epsilon, \epsilon]^{k+1}$ for the first summand, and after replacing $\mathrm{d}\theta^k$ by $(-1)^n \delta\theta^{k+1}$ and evaluating it via the maps ϱ_l in the second summand.

Applying Stokes theorem to the first summand of LHS we see that it matches the first summand of RHS except by the term

$$(-1)^n \sum_{i=0}^{p} \int_{[-\epsilon,\epsilon]^{k+1}} \theta^{k+1} \left(\frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_1}, \ldots, \frac{\mathrm{d}\mathbf{\Lambda}_i}{\mathrm{d}s_{k+1}} \right) \Bigg|_{t=\alpha_{i-1}}^{t=\alpha_i} \mathrm{d}\vec{s}.$$

The second summand of RHS matches the second summand of LHS except by the terms

$$(-1)^{m+l+3n-1} \sum_{i=0}^{p} \int_{[-\epsilon,\epsilon]^{k+1}} \left\langle \theta^{k+1}; \varrho_l \left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}s_1}(\alpha_i) \right), \ldots, \varrho_l \left(\vartheta_m \frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}s_{k+1}}(\alpha_i) \right) \right\rangle \mathrm{d}\vec{s}$$

when $l = 0$, $m = 0$ and $l = n - k$, $m = n - k - 1$. It is not difficult to see now that these last two formulas match. Hence proving that if the tuple $(\omega, \theta^1, \ldots, \theta^n)$ is a cocycle it implies that $(F, \Delta^1, \ldots, \Delta^{n-1})$ is also a cocycle. □

Then we can conclude with the following statement:

Theorem 6.2. *There is a natural cochain map τ of degree -1* (*the transgression map*)

$$\tau : \check{C}^*(\mathsf{G}, \mathbb{C}^\times(n+1)_{\mathsf{G}}) \to \check{C}^{*-1}(\mathsf{LG}, \mathbb{C}^\times(n)_{\mathsf{LG}})$$

that for $* = n$ sends gerbes to gerbes, and induces a map in Deligne cohomology

$$\mathbb{H}^* \left(\mathsf{G}, \mathbb{C}^\times(n+1)_{\mathsf{G}} \right) \to \mathbb{H}^{*-1} \left(\mathsf{LG}, \mathbb{C}^\times(n)_{\mathsf{LG}} \right).$$

Note that for $* \neq n$ we get the topological transgression map (see Proposition 2.17), i.e.

$$H^*(\mathsf{G}, \mathbb{Z}) \to H^{*-1}(\mathsf{LG}, \mathbb{Z}) \quad \text{for } * > n, \qquad H^*(\mathsf{G}, \mathbb{R}/\mathbb{Z}) \to H^{*-1}(\mathsf{LG}, \mathbb{C}^\times) \quad \text{for } * < n$$

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